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TOPICS IN MITIGATING RADAR BIAS

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14. ABSTRACT
This report considers two topics in mitigating radar bias in target tracking. The first topic considers track estimates provided by two tracking systems of a single object. There is a relative bias, computed by the method of Levedahl or Brown, Weisman and Brock. Absolute bias for each system is computed by setting up and solving the estimation problem by a minimization approach. The second topic is the use of a reduced state filter designed to provide a least mean square state estimate while mitigating measurement bias. These two topics can be used together to address specific tracking problems of interest to the Navy.

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FOREWORD

This report presents an investigation of two topics related to mitigating the effects of radar bias in tracking applications including ballistic tracking. We determine the absolute bias between two radars in polar coordinates when their relative bias is given in rectangular coordinates. Using this result, the optimized steady-state filter to handle the unknown deterministic bias is then obtained.

An earlier version of this report was published at the 2008 National Fire Control Symposium, Lexington, Massachusetts.

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NOMENCLATURE

$x(k)$	State, n-dimensional, (3-1)[1]
$\widehat{x}(k+1\|k)$	Time updated state at k, (3-3)
$\widehat{x}(k\|k)$	Measurement updated state at k, (3-4)
$z(k)$	Measurement at k, q-dimensional, (3-2)
$m(k)$	Process noise at k, n-dimensional, (3-1)
$v(k)$	Measurement noise at k, q-dimensional, (3-2)
$\Phi(l,k)$	State transition matrix from k to l, n by n-dimensional, 3-1[2]
$B(k)$	Noise input matrix at k, n by b-dimensional, 3-1
$H(k)$	Output matrix at k, q by n-dimensional, 3-1
$Q(k)$	Process noise intensity at k, n by n-dimensional, 3-1
$R(k)$	Measurement noise intensity at k, q by q-dimensional, 3-1
λ	Bias, p-dimensional, (3-2)
$\overline{\lambda}, \Lambda$	Mean and covariance of λ, 3-1
u	Bias function, $\Re^n \times \Re^p$ to $\Re^r$, (3-2)
$W(k)$	Bias matrix, q, r-dimensional, (3-2)
$\varepsilon(k\|k)$	Total error, (3-5)
$\varepsilon^{(1)}(k\|k)$	Error due to noise, (3-18)
$\varepsilon^{(2)}(k\|k)$	Error due to bias, (3-19)
$S(k\|k), S(k+1\|k)$	Total error covariances, 3-5
$M(k\|k), M(k+1\|k)$	$\varepsilon^{(1)}$ covariances, (2-20) and (3-21)
$K(k)$	Kalman filter gain matrix, n by q-dimensional, (3-4) and (3-34)
α, β	Steady state position and velocity Kalman filter gains, (3-49)
ρ	Target maneuver index, 3-22

[1] Indicates equation number.
[2] Indicates page number.

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1 INTRODUCTION

There are several facets to the problem of tracking missiles that engage in deterministic maneuvers with radar. Deterministic maneuvers can be accounted for with bias estimation techniques that enable enhanced error correction to be applied to tracking algorithms, which leads to more effectively tracked threats. In this report, we obtain the exact form of the bias error for the coordinate transformation problem. This result is useful in Ballistic Missile Defense bistatic applications where one sensor is used for launching an interceptor, while another is used to track the threat. Thus, it is important to address the problem of translation between internal sensor coordinate frames to a common frame that is used by all sensors. The coordinate transformation problem from Cartesian to spherical coordinates introduces a bias that, if accounted for, can be corrected in the filter design. This transformation problem occurs when there are multiple launch platforms, as each local track must be formatted for a common reference frame. When bias correction is accomplished correctly, one can improve the tracking performance of the filter and increase the likelihood that an interceptor can successfully engage a threat.

The results presented in this report are original and are due to the authors. An earlier version of this report appeared in [10], which is also archived in www.arXiv.org.

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2 AN OPTIMIZED METHOD OF OBTAINING ABSOLUTE BIAS

Although a relative bias calculation can be used to provide the correct association of tracks from two sensors, calculation of the absolute bias is required to correct the track state and is needed for both track fusion and producing a Single Integrated Air Picture. Levedahl [4] and Brown, Weisman, and Brock [5] present methods for obtaining the relative bias between two radars tracking the same ballistic missile, which include maximizing a likelihood function. The relative biases obtained in these papers are determined in rectangular coordinates. In this report, the absolute bias for the two sensors is calculated from the relative bias by solving a minimization problem. The absolute biases of the two sensors are given in spherical coordinates. The problem is set up to minimize the weighted sum of the two absolute biases while viewing the given relative bias as a constraint.

A point in three-dimensional space in both rectangular and spherical coordinates[1] is denoted by

$$\overrightarrow{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \overrightarrow{\pi} = \begin{bmatrix} r \\ \psi \\ \theta \end{bmatrix}, \text{ respectively.} \tag{2-1}$$

The transformations between the coordinates are $\overrightarrow{p} = f(\overrightarrow{\pi})$ and $\overrightarrow{\pi} = f^{-1}(\overrightarrow{p})$, which are given by

$$f(\overrightarrow{\pi}) = \begin{bmatrix} r\cos\theta\cos\psi \\ r\cos\theta\sin\psi \\ r\sin\theta \end{bmatrix} \text{ and } f^{-1}(\overrightarrow{p}) = \begin{bmatrix} \sqrt{x^2+y^2+z^2} \\ \arctan(y/x) \\ \arctan\left(z/\sqrt{x^2+y^2}\right) \end{bmatrix}. \tag{2-2}$$

We need the following definitions:

P_1	Target position as seen by sensor 1
P_2	Target position as seen by sensor 2
P_T	True target position (unknown)
B_1	Sensor 1 bias
B_2	Sensor 2 bias
B_R	Relative bias
P_{1TO2}	Sensor 2 position from sensor 1

Let ENU denote the East North Up coordinate system. We have $P_{1,ENU(1)} = (x_1, y_1, z_1)'_{ENU(1)}$, $P_{2,ENU(2)} = (x_2, y_2, z_2)'_{ENU(2)}$, $B_{1,ENU(1)} = (\Delta x_1, \Delta y_1, \Delta z_1)'_{ENU(1)}$, and $B_{2,ENU(2)} = (\Delta x_2, \Delta y_2, \Delta z_2)'_{ENU(2)}$. Thus, in the sensor coordinates

$$P_{T,ENU(1)} = P_{1,ENU(1)} + B_{1,ENU(1)} \tag{2-3}$$

$$P_{T,ENU(2)} = P_{2,ENU(2)} + B_{2,ENU(2)}\ . \tag{2-4}$$

[1]Denote yaw (azimuth) by ψ, pitch (elevation) by θ. ϕ is normally reserved for roll; however, roll is not used here.

If we use an ENU coordinate system located at sensor 1, (2-4) becomes[2]

$$P_{T,ENU(1)} = P_{1TO2,ENU(1)} + P_{2,ENU(1)} + B_{2,ENU(1)} \tag{2-5}$$

where $P_{1TO2,ENU(1)}$ is the position vector from the first sensor to the second sensor in ENU(1). The relative bias in ENU(1) is

$$\begin{aligned} B_{R,ENU(1)} &= B_{2,ENU(1)} - B_{1,ENU(1)} \\ &= (P_{T,ENU(1)} - P_{1TO2,ENU(1)} - P_{2,ENU(1)}) - (P_{T,ENU(1)} - P_{1,ENU(1)}) \\ &= P_{1,ENU(1)} - P_{1TO2,ENU(1)} - P_{2,ENU(1)} \ . \end{aligned} \tag{2-6}$$

We consider the coordinate transformations to allow us to go from ENU to radar-face coordinates for a particular sensor. Each sensor has its own face and ENU coordinate systems. A sensor's face coordinate system (FACE) is related to the ENU coordinate system of a sensor by the following transformation:

$$\begin{aligned} T_{ENU(i)2FACE(i)} &= \begin{bmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix} \begin{bmatrix} \cos\psi_i & \sin\psi_i & 0 \\ -\sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_i\cos\psi_i & \cos\theta_i\sin\psi_i & \sin\theta_i \\ -\sin\psi_i & \cos\psi_i & 0 \\ -\sin\theta_i\cos\psi_i & -\sin\theta_i\sin\psi_i & \cos\theta_i \end{bmatrix} \end{aligned} \tag{2-7}$$

where $i = 1, 2$. We also have that

$$T_{FACE(i)2ENU(i)} = \begin{bmatrix} \cos\theta_i\cos\psi_i & -\sin\psi_i & -\sin\theta_i\cos\psi_i \\ \cos\theta_i\sin\psi_i & \cos\psi_i & -\sin\theta_i\sin\psi_i \\ \sin\theta_i & 0 & \cos\theta_i \end{bmatrix} , \tag{2-8}$$

which is the transpose of (2-7). We can also have the matrix $T_{ENU(i)2FACE(j)}$, which is

$$T_{ENU(i)2FACE(j)} = \begin{bmatrix} \cos\theta_{i,j}\cos\psi_{i,j} & \cos\theta_{i,j}\sin\psi_{i,j} & \sin\theta_{i,j} \\ -\sin\psi_{i,j} & \cos\psi_{i,j} & 0 \\ -\sin\theta_{i,j}\cos\psi_{i,j} & -\sin\theta_{i,j}\sin\psi_{i,j} & \cos\theta_{i,j} \end{bmatrix} . \tag{2-9}$$

The absolute, as opposed to relative, bias can be expressed in the FACE coordinates:

$$B_{i,FACE(i)} = \Delta r \cdot \overrightarrow{u_r} + \Delta c_A \cdot \overrightarrow{u_{cA}} + \Delta c_B \cdot \overrightarrow{u_{cB}} \tag{2-10}$$

where $\overrightarrow{u_r}$ is the unit vector in the range coordinate and $\overrightarrow{u_{cA}}$, $\overrightarrow{u_{cB}}$ are the two cross range coordinate unit vectors. Substituting $p_T\Delta\psi = \Delta c_A$ and $p_T\Delta\theta = \Delta c_B$ where $p_{Ti} = \|P_T(i)\|$ (see note[3]), the distance from the sensor to the target, we get

$$B_{i,FACE(i)} = \Delta r_i \cdot \overrightarrow{u_r} + p_{Ti}\Delta\psi_i \cdot \overrightarrow{u_{cA}} + p_{Ti}\Delta\theta_i \cdot \overrightarrow{u_{cB}} \tag{2-11}$$

[2] Appendix A gives the transformation from ENU(1) to ENU(2).

[3] True position is not available; therefore, measured position is used when applying this method to this calculation.

$$B_{i,ENU(i)} = \begin{bmatrix} \cos\theta_i \cos\psi_i & -\sin\psi_i & -\sin\theta_i \cos\psi_i \\ \cos\theta_i \sin\psi_i & \cos\psi_i & -\sin\theta_i \sin\psi_i \\ \sin\theta_i & 0 & \cos\theta_i \end{bmatrix} \begin{bmatrix} \Delta r_i \\ p_{Ti}\Delta\psi_i \\ p_{Ti}\Delta\theta_i \end{bmatrix}$$

$$= \begin{bmatrix} \Delta r_i \cos\theta_i \cos\psi_i - \Delta\psi_i \left(\sin\psi_i\right) \cdot p_{Ti} - \Delta\theta_i \left(\sin\theta_i \cos\psi_i\right) \cdot p_{Ti} \\ \Delta r_i \cos\theta_i \sin\psi_i + \Delta\psi_i \left(\cos\psi_i\right) \cdot p_{Ti} - \Delta\theta_i \left(\sin\theta_i \sin\psi_i\right) \cdot p_{Ti} \\ \Delta r_i \sin\theta_i + \Delta\theta_i \left(\cos\theta_i\right) \cdot p_{Ti} \end{bmatrix} . \qquad (2\text{-}12)$$

We can obviously obtain $B_{i,ENU(j)}$ (for i not necessarily equal to j) if needed. The quantities Δr_1, $\Delta\psi_1$, $\Delta\theta_1$, Δr_2, $\Delta\psi_2$, $\Delta\theta_2$ are minimized. When referring to these terms as a group we write $e = (\Delta r_1, \Delta\psi_1, \Delta\theta_1, \Delta r_2, \Delta\psi_2, \Delta\theta_2)$.

We need tolerances or costs for the sensor biases. These costs are expressed in spherical coordinates.

k_{r1}	Sensor 1 range bias cost, unitless
k_{r2}	Sensor 2 range bias cost, unitless
$k_{\psi 1}$	Sensor 1 azimuth bias cost, meters
$k_{\psi 2}$	Sensor 2 azimuth bias cost, meters
$k_{\theta 1}$	Sensor 1 elevation bias cost, meters
$k_{\theta 2}$	Sensor 2 elevation bias cost, meters

Problem Statement

We want to compute the minimum (absolute) bias cost for the two sensors when there are known (computed) expressions for the relative bias. The given relative bias is expressed in ENU rectangular coordinates. We compute the minimum absolute bias in spherical coordinates. The relative bias in rectangular coordinates contrasted with the absolute bias in spherical coordinates allows us to formulate this as a minimization problem. We view the relative bias as a constraint. We use a quadratic cost:

$$F = \frac{k_{r_1}^2}{2} \cdot (\Delta r_1)^2 + \frac{k_{\psi_1}^2}{2} \cdot (\Delta\phi_1)^2 + \frac{k_{\theta_1}^2}{2} \cdot (\Delta\theta_1)^2 + \frac{k_{r_2}^2}{2} \cdot (\Delta r_2)^2 + \frac{k_{\psi_2}^2}{2} \cdot (\Delta\phi_2)^2 + \frac{k_{\theta_2}^2}{2} \cdot (\Delta\theta_2)^2 \ . \qquad (2\text{-}13)$$

So that the addition in (2-13) is permissible, we have that k_{r_1} and k_{r_2} are unitless and k_{ψ_1}, k_{θ_1}, k_{ψ_2}, and k_{θ_2} are in meters. We note F may be rewritten in the form

$$F = \begin{bmatrix} \Delta r_1 & \Delta\psi_1 & \Delta\theta_1 \end{bmatrix} \begin{bmatrix} 2/k_{r_1}^2 & 0 & 0 \\ 0 & 2/k_{\psi_1}^2 & 0 \\ 0 & 0 & 2/k_{\theta_1}^2 \end{bmatrix}^{-1} \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix}$$

$$+ \begin{bmatrix} \Delta r_2 & \Delta\psi_2 & \Delta\theta_2 \end{bmatrix} \begin{bmatrix} 2/k_{r_2}^2 & 0 & 0 \\ 0 & 2/k_{\psi_2}^2 & 0 \\ 0 & 0 & 2/k_{\theta_2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} , \qquad (2\text{-}14)$$

which we recognize as being in the form of a Mahalanobis distance. The works by Levedahl [4] and the Lincoln Laboratory [5] include the Mahalanobis distance when the log is taken of the Gaussian distribution. The cost F is minimized subject to the equality constraint

$$G(B) = \left(B_{2,ENU(1)} - B_{1,ENU(1)}\right) - B_{R,ENU(1)} = 0 \ . \qquad (2\text{-}15)$$

Thus,

$$G(B) = \begin{bmatrix} \Delta r_2 \cos\theta_2 \cos\psi_2 - \Delta\psi_2 (\sin\psi_2) \cdot p_{T2} - \Delta\theta_2 (\sin\theta_2 \cos\psi_2) \cdot p_{T2} \\ \Delta r_2 \cos\theta_2 \sin\psi_2 + \Delta\psi_2 (\cos\psi_2) \cdot p_{T2} - \Delta\theta_2 (\sin\theta_2 \sin\psi_2) \cdot p_{T2} \\ \Delta r_2 \sin\theta_2 + \Delta\theta_2 (\cos\theta_2) \cdot p_{T2} \end{bmatrix}$$

$$- \begin{bmatrix} \Delta r_1 \cos\theta_1 \cos\psi_1 - \Delta\psi_1 (\sin\psi_1) \cdot p_{T1} - \Delta\theta_1 (\sin\theta_1 \cos\psi_1) \cdot p_{T1} \\ \Delta r_1 \cos\theta_1 \sin\psi_1 + \Delta\psi_1 (\cos\psi_1) \cdot p_{T1} - \Delta\theta_1 (\sin\theta_1 \sin\psi_1) \cdot p_{T1} \\ \Delta r_1 \sin\theta_1 + \Delta\theta_1 (\cos\theta_1) \cdot p_{T1} \end{bmatrix} - B_R = 0 \qquad \text{(2-16)}$$

where all the terms in (2-16) reside entirely in one of the two ENU coordinate systems. We see that $G(B)$ gives that the difference between the two absolute biases, whatever they may be, is equal to the relative bias. Also, note that (2-16) is affine. Another equivalent representation for $G(B)$ is

$$G(B) = A(p_{T2}, \psi_2, \theta_2) \begin{bmatrix} \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} - A(p_{T1}, \psi_1, \theta_1) \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} - B_R \qquad \text{(2-17)}$$

where

$$A(p_{T1}, \psi_1, \theta_1) = \begin{bmatrix} \cos\theta_1 \cos\psi_1 & -\sin\psi_1 \cdot p_{T1} & -\sin\theta_1 \cos\psi_1 \cdot p_{T1} \\ \cos\theta_1 \sin\psi_1 & \cos\psi_1 \cdot p_{T1} & -\sin\theta_1 \sin\psi_1 \cdot p_{T1} \\ \sin\theta_1 & 0 & \cos\theta_1 \cdot p_{T1} \end{bmatrix} \qquad \text{(2-18)}$$

and

$$A(p_{T2}, \psi_2, \theta_2) = \begin{bmatrix} \cos\theta_2 \cos\psi_2 & -\sin\psi_2 \cdot p_{T2} & -\sin\theta_2 \cos\psi_2 \cdot p_{T2} \\ \cos\theta_2 \sin\psi_2 & \cos\psi_2 \cdot p_{T2} & -\sin\theta_2 \sin\psi_2 \cdot p_{T2} \\ \sin\theta_2 & 0 & \cos\theta_2 \cdot p_{T2} \end{bmatrix}. \qquad \text{(2-19)}$$

Setting $G(B) = 0$, we solve for $\Delta r_2, \Delta\psi_2, \Delta\theta_2$ using

$$\begin{bmatrix} \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} = A^{-1}(p_{T2}, \psi_2, \theta_2) \left(A(p_{T1}, \psi_1, \theta_1) \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} + B_R \right) \qquad \text{(2-20)}$$

(provided $p_{T2} \neq 0$). The vector equality constraint (2-16) can be written in the form of three scalar equality constraints.

$$G_E(B) = \Delta r_2 \cos\theta_2 \cos\psi_2 - \Delta\psi_2 \sin\psi_2 \cdot p_{T2} - \Delta\theta_2 \sin\theta_2 \cos\psi_2 \cdot p_{T2}$$

$$-\Delta r_1 \cos\theta_1 \cos\psi_1 + \Delta\psi_1 \sin\psi_1 \cdot p_{T1} + \Delta\theta_1 \sin\theta_1 \cos\psi_1 \cdot p_{T1} - B_{RE}\ , \qquad \text{(2-21)}$$

$$G_N(B) = \Delta r_2 \cos\theta_2 \sin\psi_2 + \Delta\psi_2 \cos\psi_2 \cdot p_{T2} - \Delta\theta_2 \sin\theta_2 \sin\psi_2 \cdot p_{T2}$$

$$-\Delta r_1 \cos\theta_1 \sin\psi_1 - \Delta\psi_1 \cos\psi_1 \cdot p_{T1} + \Delta\theta_1 \sin\theta_1 \sin\psi_1 \cdot p_{T1} - B_{RN}\ , \text{ and} \qquad \text{(2-22)}$$

$$G_U(B) = \Delta r_2 \sin\theta_2 + \Delta\theta_2 \cos\theta_2 \cdot p_{T2} - \Delta r_1 \sin\theta_1 - \Delta\theta_1 \cos\theta_1 \cdot p_{T1} - B_{RU}\ . \qquad \text{(2-23)}$$

Solving the Minimization Problem

To solve this minimization problem, we need to take a few derivatives. We need the gradient of the function that is minimized. We also need the gradient of the constraint, which is an equality constraint in this case.

$$\nabla F = \begin{bmatrix} \partial F/\partial \Delta r_1 \\ \partial F/\partial \Delta \psi_1 \\ \partial F/\partial \Delta \theta_1 \\ \partial F/\partial \Delta r_2 \\ \partial F/\partial \Delta \psi_2 \\ \partial F/\partial \Delta \theta_2 \end{bmatrix} = \begin{bmatrix} k_{r_1}^2 \cdot \Delta r_1 \\ k_{\psi_1}^2 \cdot \Delta \psi_1 \\ k_{\theta_1}^2 \cdot \Delta \theta_1 \\ k_{r_2}^2 \cdot \Delta r_2 \\ k_{\psi_2}^2 \cdot \Delta \psi_2 \\ k_{\theta_2}^2 \cdot \Delta \theta_2 \end{bmatrix} \tag{2-24}$$

$$\nabla G_E = \begin{bmatrix} \partial G_E/\partial \Delta r_1 \\ \partial G_E/\partial \Delta \psi_1 \\ \partial G_E/\partial \Delta \theta_1 \\ \partial G_E/\partial \Delta r_2 \\ \partial G_E/\partial \Delta \psi_2 \\ \partial G_E/\partial \Delta \theta_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_1 \cos\psi_1 \\ \sin\psi_1 \cdot p_{T1} \\ \sin\theta_1 \cos\psi_1 \cdot p_{T1} \\ \cos\theta_2 \cos\psi_2 \\ -\sin\psi_2 \cdot p_{T2} \\ -\sin\theta_2 \cos\psi_2 \cdot p_{T2} \end{bmatrix} \tag{2-25}$$

$$\nabla G_N = \begin{bmatrix} \partial G_N/\partial \Delta r_1 \\ \partial G_N/\partial \Delta \psi_1 \\ \partial G_N/\partial \Delta \theta_1 \\ \partial G_N/\partial \Delta r_2 \\ \partial G_N/\partial \Delta \psi_2 \\ \partial G_N/\partial \Delta \theta_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_1 \sin\psi_1 \\ -\cos\psi_1 \cdot p_{T1} \\ \sin\theta_1 \sin\psi_1 \cdot p_{T1} \\ \cos\theta_2 \sin\psi_2 \\ \cos\psi_2 \cdot p_{T2} \\ -\sin\theta_2 \sin\psi_2 \cdot p_{T2} \end{bmatrix} \tag{2-26}$$

$$\nabla G_U = \begin{bmatrix} \partial G_U/\partial \Delta r_1 \\ \partial G_U/\partial \Delta \psi_1 \\ \partial G_U/\partial \Delta \theta_1 \\ \partial G_U/\partial \Delta r_2 \\ \partial G_U/\partial \Delta \psi_2 \\ \partial G_u/\partial \Delta \theta_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 \\ 0 \\ -p_{T1} \cdot \cos\theta_1 \\ \sin\theta_2 \\ 0 \\ p_{T2} \cdot \cos\theta_2 \end{bmatrix} . \tag{2-27}$$

We are looking for an optimal solution located at the point $e^* = (\Delta r_1^*, \Delta\psi_1^*, \Delta\theta_1^*, \Delta r_2^*, \Delta\psi_2^*, \Delta\theta_2^*)$. We employ the Kuhn-Tucker conditions that stipulate the optimal solution e^* should satisfy these equality constraints for e, and there exist numbers a_1^*, a_2^*, a_3^* such that

$$\nabla F(e^*) = a_1^* \cdot \nabla G_E(e^*) + a_2^* \cdot \nabla G_N(e^*) + a_3^* \cdot \nabla G_U(e^*) \ . \tag{2-28}$$

The gradients $\nabla G_E(e^*), \nabla G_N(e^*), \nabla G_U(e^*)$ are linearly independent. Taking an inventory of the equations and unknowns, we see that there are nine unknowns (e, a_1, a_2, a_3) and nine equations [three from the equality constraint and six from (2-28)]. We *may* be able to find the solution. Since the cost F is quadratic and the constraint G is affine, the necessary conditions we give for optimality are also sufficient conditions, and an optimal solution e^* is a global optimal solution.

Equation (2-28) in longhand is:

$$\begin{bmatrix} k_{r_1}^2 \cdot \Delta r_1 \\ k_{\psi_1}^2 \cdot \Delta\psi_1 \\ k_{\theta_1}^2 \cdot \Delta\theta_1 \\ k_{r_2}^2 \cdot \Delta r_2 \\ k_{\psi_2}^2 \cdot \Delta\psi_2 \\ k_{\theta_2}^2 \cdot \Delta\theta_2 \end{bmatrix} = a_1 \begin{bmatrix} -\cos\theta\cos\psi_1 \\ \sin\psi_1 \cdot p_{T1} \\ \sin\theta_1\cos\psi_1 \cdot p_{T1} \\ \cos\theta_2\cos\psi_2 \\ -\sin\psi_2 \cdot p_{T2} \\ -\sin\theta_2\cos\psi_2 \cdot p_{T2} \end{bmatrix} + a_2 \begin{bmatrix} -\cos\theta_1\sin\psi_1 \\ -\cos\psi_1 \cdot p_{T1} \\ \sin\theta_1\sin\psi_1 \cdot p_{T1} \\ \cos\theta_2\sin\psi_2 \\ \cos\psi_2 \cdot p_{T2} \\ -\sin\theta_2\sin\psi_2 \cdot p_{T1} \end{bmatrix}$$

$$+a_3 \begin{bmatrix} -\sin\theta_1 \\ 0 \\ -p_{T1} \cdot \cos\theta_1 \\ \sin\theta_2 \\ 0 \\ p_{T2} \cdot \cos\theta_2 \end{bmatrix} . \tag{2-29}$$

The right hand side of (2-29) may be written in the form of the product of two matrices, M_1 and M_2.

$$\begin{bmatrix} -\cos\theta\cos\psi_1 & -\cos\theta_1\sin\psi_1 & -\sin\theta_1 \\ \sin\psi_1 \cdot p_{T1} & -\cos\psi_1 \cdot p_{T1} & 0 \\ \sin\theta_1\cos\psi_1 \cdot p_{T1} & \sin\theta_1\sin\psi_1 \cdot p_{T1} & -p_{T1} \cdot \cos\theta_1 \\ \cos\theta_2\cos\psi_2 & \cos\theta_2\sin\psi_2 & \sin\theta_2 \\ -\sin\psi_2 \cdot p_{T2} & \cos\psi_2 \cdot p_{T2} & 0 \\ -\sin\theta_2\cos\psi_2 \cdot p_{T2} & -\sin\theta_2\sin\psi_2 \cdot p_{T2} & p_{T2} \cdot \cos\theta_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} , \tag{2-30}$$

where

$$M_1 = \begin{bmatrix} -\cos\theta\cos\psi_1 & -\cos\theta_1\sin\psi_1 & -\sin\theta_1 \\ \sin\psi_1 \cdot p_{T1} & -\cos\psi_1 \cdot p_T & 0 \\ \sin\theta_1\cos\psi_1 \cdot p_{T1} & \sin\theta_1\sin\psi_1 \cdot p_T & -p_{T1} \cdot \cos\theta_1 \end{bmatrix} \tag{2-31}$$

and

$$M_2 = \begin{bmatrix} \cos\theta_2\cos\psi_2 & \cos\theta_2\sin\psi_2 & \sin\theta_2 \\ -\sin\psi_2 \cdot p_{T1} & \cos\psi_2 \cdot p_{2T} & 0 \\ -\sin\theta_2\cos\psi_2 \cdot p_{T1} & -\sin\theta_2\sin\psi_2 \cdot p_{2T} & p_{T2} \cdot \cos\theta_2 \end{bmatrix} . \tag{2-32}$$

Note that a_1, a_2, and a_3 have the units of meters. Let

$$D_1 = \begin{bmatrix} k_{r_1}^2 & 0 & 0 \\ 0 & k_{\psi_1}^2 & 0 \\ 0 & 0 & k_{\theta_1}^2 \end{bmatrix} \tag{2-33}$$

$$D_2 = \begin{bmatrix} k_{r_2}^2 & 0 & 0 \\ 0 & k_{\psi_2}^2 & 0 \\ 0 & 0 & k_{\theta_2}^2 \end{bmatrix} . \tag{2-34}$$

Rewriting the left hand side of (2-29) as

$$\begin{bmatrix} k_{r_1}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{\psi_1}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{\theta_1}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{r_2}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{\psi_2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{\theta_2}^2 \end{bmatrix} \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \\ \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0_{3,3} \\ 0_{3,3} & D_2 \end{bmatrix} \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \\ \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} , \tag{2-35}$$

we have

$$D_1 \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} = M_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} , \tag{2-36}$$

$$D_2 \begin{bmatrix} \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} = M_2 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} , \tag{2-37}$$

or,

$$\begin{bmatrix} \Delta r_2 \\ \Delta\psi_2 \\ \Delta\theta_2 \end{bmatrix} = D_2^{-1} M_2 M_1^{-1} D_1 \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} . \tag{2-38}$$

Substituting (2-38) into (2-20) yields

$$D_2^{-1} M_2 M_1^{-1} D_1 \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} = A^{-1}\left(p_{T2}, \psi_2, \theta_2\right) \left(A\left(p_{T1}, \psi_1, \theta_1\right) \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} + B_R \right) , \tag{2-39}$$

so we get

$$\left(D_2^{-1} M_2 M_1^{-1} D_1 - A^{-1}\left(p_{T2}, \psi_2, \theta_2\right) A\left(p_{T1}, \psi_1, \theta_1\right) \right) \begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} = A^{-1}\left(p_{T2}, \psi_2, \theta_2\right) B_R \tag{2-40}$$

$$\begin{bmatrix} \Delta r_1 \\ \Delta\psi_1 \\ \Delta\theta_1 \end{bmatrix} = \left(D_2^{-1} M_2 M_1^{-1} D_1 - A^{-1}\left(p_{T2}, \psi_2, \theta_2\right) A\left(p_{T1}, \psi_1, \theta_1\right) \right)^{-1} A^{-1}\left(p_{T2}, \psi_2, \theta_2\right) B_R , \tag{2-41}$$

which allows us to obtain $(\Delta r_1, \Delta\psi_1, \Delta\theta_1)$. Finally, substituting (2-41) into (2-38) we get $(\Delta r_2, \Delta\psi_2, \Delta\theta_2)$.

Numerical Examples

The examples below illustrate this idea.

Example A

INPUT	OUTPUT
B_R = [200 500 300]'	Cost = 1.6250e+004
$p_{T1} = 25000$	Δr_1 = -1.7678e+002
$\psi_1 = 0$	$\Delta\psi_1$ = -1.0000e-002
θ_1 = 7.8540e-001	$\Delta\theta_1$ = -1.4142e-003
$p_{T2} = 50000$	Δr_2 = 3.5355e+001
$\psi_2 = 0$	$\Delta\psi_2$ = 5.0000e-003
θ_2 = 2.3562e+000	$\Delta\theta_2$ = -3.5355e-003
$k_{r1}^2 = 2$	
$k_{\psi 1}^2$ = 1.2500e+009 $= 2 * \mathrm{PT1}^2$	
$k_{\theta 1}^2$ = 1.2500e+009	
$k_{r2}^2 = 2$	
$k_{\psi 2}^2$ = 5.0000e+009 $= 2 * \mathrm{PT2}^2$	
$k_{\theta 2}^2$ = 5.0000e+009	

Example B

INPUT Same as Example A but with	OUTPUT
B_R = [200 0 500]'	Cost = 3.6250e+004
	Δr_1 = -2.4749e+002
	$\Delta\psi_1 = 0$
	$\Delta\theta_1$ = -4.2426e-003
	Δr_2 = 1.0607e+002
	$\Delta\psi_2 = 0$
	$\Delta\theta_2$ = -4.9497e-003

Example C

INPUT Same as Example A but with	OUTPUT
$\psi_2 = \pi$	Cost = 1.6250e+004
$\theta_2 = \pi/4$	Δr_1 = -1.7678e+002
	$\Delta\psi_1$ = -1.0000e-002
	$\Delta\theta_1$ = -1.4142e-003
	Δr_2 = 3.5355e+001
	$\Delta\psi_2$ = -5.0000e-003
	$\Delta\theta_2$ = 3.5355e-003

The input for this case is a variation of the input in Example A. Note that the output for this case is the same as Example A, with the exception of a sign swap between $\Delta\psi_2$ and $\Delta\theta_2$ to account for the orientation difference of the “2” coordinates.

Example D

INPUT Same as Example A but with	OUTPUT
$\psi_2 = \pi/2$	Cost = 5.5625e+004
$\theta_2 = \pi/4$	Δr_1 = -1.7678e+002
	$\Delta\psi_1$ = -1.0000e-002
	$\Delta\theta_1$ = -1.4142e-003
	Δr_2 = 2.8284e+002
	$\Delta\psi_2$ = -2.0000e-003
	$\Delta\theta_2$ = -1.4142e-003

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3 AN OPTIMIZED REDUCED-STATE FILTER FOR UNKNOWN BIAS

Mookerjee and Reifler [7] present a novel technique for calculating a steady-state reduced-order filter to track a maneuvering target. The filter they derive is optimized for performance with a stochastic acceleration. In this chapter, this technique is modified to derive a steady-state filter optimized for performance with a stochastic measurement bias. Similar to Mookerjee and Reifler [7], the filter developed in this chapter is a reduced-state filter.

The principles of a reduced-state filter applied to bias estimation can be understood by considering [8] and [9]. In these reports, the position and velocity of an aircraft (a Beechcraft 1900) with DMEs [distance measuring equipment], an INS [inertial navigation system], and a barometric altimeter are estimated. The filter (in [8]) and the smoother (in [9]) were designed with a state-to-estimate range bias in each DME (up to 5 were used), a state-to-estimate INS drift, and a state-to-estimate bias in the barometric altimeter. The filter (or smoother) ran with these additional bias states in tow (i.e., in addition to the position and velocity states). (The results in [8] and [9] achieved the design goals in position and velocity accuracy.)

We use discrete time dynamical equations in this report. It is fair to consider the state and output (dynamical) equations to be the dual, in the control theory sense, of the state and input equations of [7]. These are (8) and (5) of [7]. Compared to the dynamical equations in Mookerjee and Reifler [7], we eliminate the unknown acceleration from the state equation and add an unknown bias in the output (measurement) equation, the typical dual situation. We have:

$$x\left(k+1\right) = \Phi\left(k+1,k\right)x\left(k\right) + B\left(k\right)m\left(k\right) \tag{3-1}$$

$$z\left(k\right) = H\left(k\right)\cdot x\left(k\right) + v\left(k\right) + W\left(k\right)u\left(x\left(k\right),\lambda\right) \ . \tag{3-2}$$

The state $x(k)$ at time k is of dimension n. The state transition matrix[4] $\Phi\left(l,k\right)$, of dimension n by n, propagates the state in time from k to l in the absence of noise. The noise input matrix $B\left(k\right)$ is of dimension n by b. The output $z(k)$ at time k is of dimension q and the output matrix $H\left(k\right)$ is of dimension q by n. The process noise term $m(k)$ is of dimension b with covariance $Q\left(k\right)$. In the sequel, x represents positions and velocities, m represents accelerations, and $B\left(k\right)$ is an adjustment matrix between position, velocity and acceleration. The measurement noise term $v(k)$ is of dimension q with covariance $R\left(k\right)$. The bias matrix $W\left(k\right)$ is q by r. The bias function u is $\Re^n \times \Re^p \rightarrow \Re^r$, and we have that the bias λ is a p-dimensional random vector with mean $\overline{\lambda}$ and covariance Λ.

The time update equation, using (3-1), is simply

$$\widehat{x}\left(k+1|k\right) = \Phi\left(k+1,k\right)\widehat{x}\left(k|k\right) \ . \tag{3-3}$$

The measurement update equation becomes

$$\begin{aligned}\widehat{x}\left(k+1|k+1\right) &= \widehat{x}\left(k+1|k\right) + K\left(k+1\right)\left\{z\left(k+1\right)\right.\\ &\left.-H\left(k+1\right)\widehat{x}\left(k+1|k\right) - W\left(k+1\right)u\left(\widehat{x}\left(k+1|k\right),\overline{\lambda}\right)\right\} \ ,\end{aligned} \tag{3-4}$$

[4] Bar-Shalom, Rong Li, and Kirubarajan [2].

where $K(k)$ is the n by q measurement, or Kalman, gain matrix. In the steady-state case, which is discussed below, the position gain α and velocity gain β substitute for $K(k)$.

Filter Development – General Case

In this section, we develop the filter equations for the general case. The development in this chapter is (basically) dual (dual in the sense of control theory) to Section III in [7]. The error is defined as (we develop the errors analogous to (27) and (32) of [7]):

$$\varepsilon(k+1|k+1) \equiv x(k+1) - \widehat{x}(k+1|k+1) \tag{3-5}$$

$$= x(k+1) - \widehat{x}(k+1|k) - K(k+1)(z(k+1) - H(k+1)\widehat{x}(k+1|k)$$
$$-W(k+1)u\left(\widehat{x}(k+1|k), \overline{\lambda}\right))$$
$$= x(k+1) - \widehat{x}(k+1|k)$$
$$-K(k+1)\{H(k+1)x(k+1) + v(k+1) + W(k+1)u(x(k+1), \lambda)$$
$$-H(k+1)\widehat{x}(k+1|k) - W(k+1)u\left(\widehat{x}(k+1|k), \overline{\lambda}\right)\} \ .$$

Continuing,

$$\varepsilon(k+1|k+1)$$
$$= x(k+1) - K(k+1)H(k+1)x(k+1) - K(k+1)v(k+1)$$
$$-K(k+1)W(k+1)u(x(k+1), \lambda)$$
$$-\widehat{x}(k+1|k) + K(k+1)H(k+1)\widehat{x}(k+1|k) + K(k+1)W(k+1)u\left(\widehat{x}(k+1|k), \overline{\lambda}\right)$$
$$= \Phi(k+1,k)x(k) + B(k)m(k) - K(k+1)H(k+1)(\Phi(k+1,k)x(k) + B(k)m(k))$$
$$-K(k+1)v(k+1) - K(k+1)W(k+1)u(\Phi(k+1,k)x(k) + B(k)m(k), \lambda)$$
$$-\Phi(k+1,k)\widehat{x}(k|k) + K(k+1)H(k+1)\Phi(k+1,k)\widehat{x}(k|k)$$
$$+K(k+1)W(k+1)u\left(\Phi(k+1,k)\widehat{x}(k|k), \overline{\lambda}\right)$$
$$= (I - K(k+1)H(k+1))\Phi(k+1,k)(x(k) - \widehat{x}(k|k))$$
$$+(I - K(k+1)H(k+1))B(k)m(k) - K(k+1)v(k+1)$$
$$-K(k+1)W(k+1)\left\{u(\Phi(k+1,k)x(k) + B(k)m(k), \lambda) - u\left(\Phi(k+1,k)\widehat{x}(k|k), \overline{\lambda}\right)\right\} \ .$$

So

$$\varepsilon(k+1|k+1) = L(k+1)\Phi(k+1,k)\varepsilon(k|k)$$
$$+L(k+1)B(k)m(k) - K(k+1)\left(W(k+1)\Delta u_{k+1|k+1} + v(k+1)\right) \ ; \tag{3-6}$$

where

$$L(k) = (I - K(k)H(k)) \tag{3-7}$$

an n by n matrix, and

$$\Delta u_{k+1|k+1} \equiv u(\Phi(k+1,k)x(k) + B(k)m(k), \lambda) - u\left(\Phi(k+1,k)\widehat{x}(k|k), \overline{\lambda}\right) \ . \tag{3-8}$$

We make the linear approximation

$$\Delta u_{k+1|k+1} \approx \left.\frac{\partial u}{\partial x}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}} \left(\Phi\left(k+1,k\right)\Delta x + B\left(k\right)m\left(k\right)\right)$$
$$+ \left.\frac{\partial u}{\partial \lambda}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}} \Delta\lambda \tag{3-9}$$

where

$$\Delta x = \varepsilon\left(k|k\right) = x\left(k\right) - \widehat{x}\left(k|k\right)\ , \tag{3-10}$$

and

$$\Delta\lambda = \lambda - \overline{\lambda}\ . \tag{3-11}$$

The result obtained is

$$\begin{aligned}
\varepsilon\left(k+1|k+1\right) &= L\left(k+1\right)\Phi\left(k+1,k\right)\varepsilon\left(k|k\right) + L\left(k+1\right)B\left(k\right)m\left(k\right) \\
&\quad - K\left(k+1\right)W\left(k+1\right) \\
&\quad \times \left(\left.\frac{\partial u}{\partial x}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}} \left(\Phi\left(k+1,k\right)\varepsilon\left(k|k\right) + B\left(k\right)m\left(k\right)\right) + \left.\frac{\partial u}{\partial \lambda}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}} \Delta\lambda\right) \\
&\quad - K\left(k+1\right)v\left(k+1\right) \\
&= \left(L\left(k+1\right) - K\left(k+1\right)W\left(k+1\right)\left.\frac{\partial u}{\partial x}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}}\right)\Phi\left(k+1,k\right)\varepsilon\left(k|k\right) \\
&\quad - K\left(k+1\right)W\left(k+1\right)\left.\frac{\partial u}{\partial \lambda}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}} \Delta\lambda \\
&\quad + \left(L\left(k+1\right) - K\left(k+1\right)W\left(k+1\right)\left.\frac{\partial u}{\partial x}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}}\right)B\left(k\right)m\left(k\right) - K\left(k+1\right)v\left(k+1\right)\ .
\end{aligned} \tag{3-12}$$

Define

$$F\left(k+1,k\right) = \left(L\left(k+1\right) - K\left(k+1\right)W\left(k+1\right)\left.\frac{\partial u}{\partial x}\right|_{x=\widehat{x}(k+1|k+1),\lambda=\overline{\lambda}}\right)\Phi\left(k+1,k\right)\ , \tag{3-13}$$

an n by n matrix, and

$$C\left(k\right) = -K\left(k\right)W\left(k\right)\left.\frac{\partial u}{\partial \lambda}\right|_{x=\widehat{x}(k|k),\lambda=\overline{\lambda}}\ , \tag{3-14}$$

an n by p matrix. Set

$$\widetilde{m}\left(k\right) = \Phi^{-1}\left(k+1,k\right)B\left(k\right)m\left(k\right) \tag{3-15}$$

and take note that the covariance of $\widetilde{m}$ is

$$\begin{aligned}
E[\widetilde{m}\left(k\right)\widetilde{m}\left(k\right)'] = \widetilde{Q}\left(k\right) &= E\left[\Phi^{-1}\left(k+1,k\right)B\left(k\right)m\left(k\right)m\left(k\right)'B\left(k\right)'\left(\Phi^{-1}\left(k+1,k\right)\right)'\right] \\
&= \Phi^{-1}\left(k+1,k\right)B\left(k\right)Q\left(k\right)B\left(k\right)'\left(\Phi^{-1}\left(k+1,k\right)\right)'\ .
\end{aligned} \tag{3-16}$$

Then (3-12) becomes

$$\varepsilon(k+1|k+1) = F(k+1,k)\,\varepsilon(k|k) + F(k+1,k)\,\widetilde{m}(k) + C(k+1)\,\Delta\lambda - K(k+1)\,v(k+1)\ . \tag{3-17}$$

We now implement the observation made in [7] that the error $\varepsilon(k|k)$ may be viewed as consisting of two components. The first component of error, $\varepsilon^{(1)}$, is due to the (unbiased) process noise $\widetilde{m}$, (3-15), and the measurement noise v. The second component of error, $\varepsilon^{(2)}$, is due to the measurement bias. To the extent that the linear approximation is valid, a linear analysis holds. That is, the two error inputs may be treated in separate equations by applying the superposition principle of linear analysis. Applying the superposition principle to (3-17) we get these two equations:

$$\varepsilon^{(1)}(k+1|k+1) = F(k+1,k)\,\varepsilon^{(1)}(k|k) + F(k+1,k)\,\widetilde{m}(k) - K(k+1)\,v(k+1) \tag{3-18}$$

$$\varepsilon^{(2)}(k+1|k+1) = F(k+1,k)\,\varepsilon^{(2)}(k|k) + C(k+1)\cdot\Delta\lambda\ . \tag{3-19}$$

These equations are comparable to (33) and (24) of [7]. Equation (3-18) contains the unbiased noise and (3-19) contains the bias.

In addition, we require update equations for the total covariance and the covariance of $\varepsilon^{(1)}(k|k)$. Define

$$M(k+1|k) \equiv E\left[\varepsilon^{(1)}(k+1|k)\,\varepsilon^{(1)}(k+1|k)'\right] \tag{3-20}$$

and

$$M(k+1|k+1) \equiv E\left[\varepsilon^{(1)}(k+1|k+1)\,\varepsilon^{(1)}(k+1|k+1)'\right]\ . \tag{3-21}$$

Substituting (3-1) into (3-20) we get that the time updated covariance for $\varepsilon^{(1)}$ is

$$\begin{aligned}
M(k+1|k) &= E\left[x(k+1) - \widehat{x}(k+1|k)\right]\left[x(k+1) - \widehat{x}(k+1|k)\right]' \\
&= E\left[\Phi(k+1,k)\,x(k) + B(k)\,m(k) - \Phi(k+1,k)\,\widehat{x}(k|k)\right] \\
&\quad \left[\Phi(k+1,k)\,x(k) + B(k)\,m(k) - \Phi(k+1,k)\,\widehat{x}(k|k)\right]' \\
&= E\left[\Phi(k+1,k)\,(x(k) - \widehat{x}(k|k)) + B(k)\,m(k)\right]\left[\Phi(k+1,k)\,(x(k) - \widehat{x}(k|k)) + B(k)\,m(k)\right]' \\
&= \Phi(k+1,k)\,M(k|k)\,\Phi(k+1,k)' + B(k)\,Q(k)\,B(k)'\ .
\end{aligned} \tag{3-22}$$

Substituting (3-18) into (3-21) measurement updated covariance for $\varepsilon^{(1)}$ is

$$\begin{aligned}
M(k+1|k+1) &= E\left[\varepsilon^{(1)}(k+1|k+1)\,\varepsilon^{(1)}(k+1|k+1)'\right] \\
&= E\Big[\Big(F(k+1,k)\,\varepsilon^{(1)}(k|k) + F(k+1,k)\,\widetilde{m}(k) - K(k+1)\,v(k+1)\Big) \\
&\quad \Big(F(k+1,k)\,\varepsilon^{(1)}(k|k) + F(k+1,k)\,\widetilde{m}(k) - K(k+1)\,v(k+1)\Big)'\Big]
\end{aligned}$$

and we use $E\left[v(k)\,v(l)'\right] = 0$ for $k \neq l$ giving
$E\left[F(k+1,k)\,\varepsilon^{(1)}(k|k)\,(K(k+1)\,v(k+1))'\right] = 0$. Hence,

$$M(k+1|k+1) = E\left[F(k+1,k)\,\varepsilon^{(1)}(k|k)\,\varepsilon^{(1)}(k|k)'\,F(k+1,k)'\right]$$

$$+E\left[F(k+1,k)\,\widetilde{m}(k)\,\widetilde{m}(k)'\,F(k+1,k)'+K(k+1)\,v(k+1)\,v(k+1)'\,K(k+1)'\right]$$
$$=F(k+1,k)\,M(k|k)\,F(k+1,k)'+F(k+1,k)\,\widetilde{Q}(k)\,F(k+1,k)'$$
$$+K(k+1)\,R(k+1)\,K(k+1)' \ . \tag{3-23}$$

Our goal is to formulate the update equations for the total covariance, so we define the n by p matrices $D(k|k)$ and $D(k+1|k)$, and then note

$$\varepsilon^{(2)}(k|k) \equiv D(k|k)\cdot\Delta\lambda \ . \tag{3-24}$$

In view of our linearized analysis, we can define $D(k|k)$ in this way because the system output ($\varepsilon^{(2)}(k|k)$) is a linear function of the system input ($\Delta\lambda$). Proceeding, $D(k+1|k)$ is defined as

$$D(k+1|k) \equiv F(k+1,k)\,D(k|k) \ . \tag{3-25}$$

In (3-24), $\varepsilon^{(2)}$ and $\Delta\lambda$ are known quantities [the equation defines $D(k|k)$]. In (3-25), $F(k)$ and $D(k|k)$ are known quantities. Then, substituting (3-24) into (3-19), we obtain

$$D(k+1|k+1)\cdot\Delta\lambda = F(k+1,k)\,D(k|k)\cdot\Delta\lambda + C(k+1)\cdot\Delta\lambda$$
$$= D(k+1|k)\cdot\Delta\lambda + C(k+1)\cdot\Delta\lambda \ , \tag{3-26}$$

and consequently (since (3-26) holds for all $\Delta\lambda$)

$$D(k+1|k+1) = D(k+1|k) + C(k+1) \ . \tag{3-27}$$

Let S be the total error (due to m, v, and λ) covariance. By superposition, we get the total error by the addition of the two error terms. We observe that since the two errors, $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$, originate from independent sources, they remain independent for all times k. By considering the definition of S (below), we observe that

$$S(k+1|k) \equiv E\left[\varepsilon(k+1|k)\,\varepsilon(k+1|k)'\right]$$
$$= E\left[\left(\varepsilon^{(1)}(k+1|k)+\varepsilon^{(2)}(k+1|k)\right)\left(\varepsilon^{(1)}(k+1|k)+\varepsilon^{(2)}(k+1|k)\right)'\right]$$
$$= E\left[\varepsilon^{(1)}(k+1|k)\,\varepsilon^{(1)}(k+1|k)'\right]+E\left[\varepsilon^{(2)}(k+1|k)\,\varepsilon^{(2)}(k+1|k)'\right]$$
$$= M(k+1|k)+E\left[\varepsilon^{(2)}(k+1|k)\,\varepsilon^{(2)}(k+1|k)'\right]$$
$$= M(k+1|k)+E\left[\Phi(k+1,k)\,D(k|k)\,\Delta\lambda\cdot\Delta\lambda' D(k|k)'\,\Phi(k+1,k)'\right] \ ,$$

using (3-1)[5], (3-3), and (3-24). Hence,

$$S(k+1|k) = M(k+1|k)+\Phi(k+1,k)\,D(k|k)\,E\left[\Delta\lambda\Delta\lambda'\right]D(k|k)'\,\Phi(k+1,k)' \ .$$

Finally, we note

$$S(k+1|k) = M(k+1|k)+\Phi(k+1,k)\,D(k|k)\,\Lambda D(k|k)'\,\Phi(k+1,k)' \ . \tag{3-28}$$

[5] The process noise part of (3-1) does not figure into $\varepsilon^{(2)}$.

Basically, this is the same result found in (19) of [7].

We next obtain the measurement update for S:

$$S(k+1|k+1) \equiv E\left[\varepsilon(k+1|k+1)\,\varepsilon(k+1|k+1)'\right]$$

$$= E\left[(x(k+1) - \widehat{x}(k+1|k+1))\,(x(k+1) - \widehat{x}(k+1|k+1))'\right]$$

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k)\right.$$

$$-K(k+1)\left(z(k+1) - H(k+1)\,\widehat{x}(k+1|k) - W(k+1)\,u\left(\widehat{x}(k+1|k),\overline{\lambda}\right)\right)\}$$

$$\left.\{\text{ditto}\}'\right] \;,$$

using (3-4),

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k)\right.$$

$$-K(k+1)\,(H(k+1)\,x(k+1) + v(k+1) + W(k+1)\,u(x(k+1),\lambda)$$

$$\left.-H(k+1)\,\widehat{x}(k+1|k) - W(k+1)\,u\left(\widehat{x}(k+1|k),\overline{\lambda}\right)\right)\}\,\{\text{ditto}\}'\right] \;,$$

using (3-2),

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k) - K(k+1)\,H(k+1)\,(x(k+1) - \widehat{x}(k+1|k))\right.$$

$$\left.-K(k+1)\left(v(k+1) + W(k+1)\,u(x(k+1),\lambda) - W(k+1)\,u\left(\widehat{x}(k+1|k),\overline{\lambda}\right)\right)\}\,\{\text{ditto}\}'\right]$$

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k) - K(k+1)\,H(k+1)\,(x(k+1) - \widehat{x}(k+1|k))\right.$$

$$\left.-K(k+1)\left(v(k+1) + W(k+1)\left(u(x(k+1),\lambda) - u\left(\widehat{x}(k+1|k),\overline{\lambda}\right)\right)\right)\}\,\{\text{ditto}\}'\right]$$

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k) - K(k+1)\,H(k+1)\,(x(k+1) - \widehat{x}(k+1|k))\right.$$

$$-K(k)\left(v(k+1) + W(k+1)\left(u(\Phi(k+1,k)\,x(k) + m(k),\lambda) - u\left(\Phi(k+1,k)\,\widehat{x}(k|k),\overline{\lambda}\right)\right)\right)\}$$

$$\left.\{\text{ditto}\}'\right]$$

$$= E\left[\{x(k+1) - \widehat{x}(k+1|k) - K(k+1)\,H(k+1)\,(x(k+1) - \widehat{x}(k+1|k))\right.$$

$$\left.-K(k+1)\left(v(k+1) + W(k+1)\left(\Delta u_{k+1|k+1}\right)\right)\}\,\{\text{ditto}\}'\;\right] \;,$$

using (3-8).

We take this next step only to the extent of the approximation,

$$S(k+1|k+1) = E\left[\{(I - K(k+1)\,H(k+1))\,(x(k+1) - \widehat{x}(k+1|k))\right.$$

$$-K(k+1)\left(v(k+1) + W(k+1)\left(\frac{\partial u}{\partial x}(\Phi(k+1,k)\,\varepsilon(k|k) + m(k)) + \frac{\partial u}{\partial \lambda}\Delta\lambda\right)\right)\Bigg\}$$

$$\left.\{\text{ditto}\}'\right] \;,$$

using (3-9), where we have omitted the substitution limits on the partial derivative fractions. Continuing,

$$= E\left[\{(I - K(k+1)\,H(k+1))\,\varepsilon(k+1|k)\right.$$

$$-K(k+1)\left(v(k+1) + W(k+1)\left(\frac{\partial u}{\partial x}\varepsilon(k+1|k) + \frac{\partial u}{\partial \lambda}\Delta\lambda\right)\right)\Bigg\}$$

$$\left.\{\text{ditto}\}'\right]$$

$$= E\left[\left\{\left(I - K(k+1)H(k+1) - K(k+1)W(k+1)\frac{\partial u}{\partial x}\right)\varepsilon(k+1|k)\right.\right.$$

$$\left.-K(k+1)\left(v(k+1) + W(k+1)\frac{\partial u}{\partial \lambda}\Delta\lambda\right)\right\}$$

$$\left.\{\text{ditto}\}'\right] \ .$$

Let

$$\widetilde{H}(k) = H(k) + W(k)\frac{\partial u}{\partial x} \tag{3-29}$$

and

$$\widetilde{R}(k) = R(k) + W(k)\frac{\partial u}{\partial \lambda}\Lambda\frac{\partial u}{\partial \lambda}'W(k)' \ . \tag{3-30}$$

Therefore, we have

$$S(k+1|k+1) = \left(I - K(k+1)\widetilde{H}(k+1)\right)S(k+1|k)\left(I - K(k+1)\widetilde{H}(k+1)\right)'$$

$$+K(k+1)\widetilde{R}(k+1)K(k+1)'$$

$$-\left(I - K(k+1)\widetilde{H}(k+1)\right)E\left[\varepsilon(k+1|k)\Delta\lambda'\right]\left(W(k+1)\frac{\partial u}{\partial \lambda}\right)'K(k+1)'$$

$$-K(k+1)\left(W(k+1)\frac{\partial u}{\partial \lambda}\right)E\left[\Delta\lambda\varepsilon(k+1|k)'\right]\left(I - K(k+1)\widetilde{H}(k+1)\right)' \ .$$

Combining (3-1) and (3-3) and substituting into (3-24) yields

$$E\left[\varepsilon(k+1|k)\Delta\lambda'\right] = E\left[\left(\varepsilon^{(1)}(k+1|k) + \varepsilon^{(2)}(k+1|k)\right)\Delta\lambda'\right]$$

$$= E\left[\varepsilon^{(2)}(k+1|k)\Delta\lambda'\right] = \Phi(k+1,k)E\left[\varepsilon^{(2)}(k|k)\Delta\lambda'\right]$$

$$= \Phi(k+1,k)E\left[D(k|k)\Delta\lambda\Delta\lambda'\right] = \Phi(k+1,k)D(k|k)\Lambda \ .$$

Hence,

$$S(k+1|k+1) = \left(I - K(k+1)\widetilde{H}(k+1)\right)S(k+1|k)\left(I - K(k+1)\widetilde{H}(k+1)\right)'$$

$$+K(k+1)\widetilde{R}(k+1)K(k+1)'$$

$$-\left(I - K(k+1)\widetilde{H}(k+1)\right)\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial \lambda}\right)'K(k+1)'$$

$$-K(k+1)\left(W(k+1)\frac{\partial u}{\partial \lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'\left(I - K(k+1)\widetilde{H}(k+1)\right)' \ . \tag{3-31}$$

Equation (3-31) is similar in form to (37) of [7]. The completing-the-square technique may be used to solve for $K(k+1)$ in this equation. First, we expand (3-31) as

$$S(k+1|k+1) = S(k+1|k) + K(k+1)\widetilde{H}(k+1)S(k+1|k)\widetilde{H}(k+1)'K(k+1)'$$

$$+K(k+1)\widetilde{R}(k+1)K(k+1)'$$

$$+K(k+1)\widetilde{H}(k+1)\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)'K(k+1)'$$

$$+K(k+1)\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'\widetilde{H}(k+1)'K(k+1)'$$

$$-\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)'K(k+1)'$$

$$-K(k+1)\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'$$

$$-K(k+1)\widetilde{H}(k+1)S(k+1|k)-S(k+1|k)\widetilde{H}(k+1)'K(k+1)' \ .$$

Next, gather the like terms:

$$S(k+1|k+1)=S(k+1|k)$$
$$+K(k+1)\Big(\widetilde{H}(k+1)S(k+1|k)\widetilde{H}(k+1)'+\widetilde{R}(k+1)$$
$$+\widetilde{H}(k+1)\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)'$$
$$+\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'\widetilde{H}(k+1)'\Big)K(k+1)'$$
$$-K(k+1)\left(\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'+\widetilde{H}(k+1)S(k+1|k)\right)$$
$$-\left(\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)'+S(k+1|k)\widetilde{H}(k+1)'\right)K(k+1)' \ . \tag{3-32}$$

To condense the notation, define

$$X(k+1)=\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'$$

$$Y(k+1)=\widetilde{H}(k+1)S(k+1|k)\widetilde{H}(k+1)'+\widetilde{R}(k+1)$$
$$+\widetilde{H}(k+1)\Phi(k+1,k)D(k|k)\Lambda\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)'$$
$$+\left(W(k+1)\frac{\partial u}{\partial\lambda}\right)\Lambda D(k|k)'\Phi(k+1,k)'\widetilde{H}(k+1)'$$
$$=\left(\widetilde{H}(k+1)S(k+1|k)\widetilde{H}(k+1)'+\widetilde{R}(k+1)+\widetilde{H}(k+1)X(k+1)'+X(k+1)\widetilde{H}(k+1)'\right)$$

and

$$Z(k+1)=\left(X(k+1)'+S(k+1|k)\widetilde{H}(k+1)'\right)Y(k+1)^{-1} \ .$$

We have that $X(k)$ is a q by n matrix, $Y(k)$ is a q by q matrix, and $Z(k)$ is an n by q matrix. Then (3-32) becomes

$$S(k+1|k+1)=S(k+1|k)+K(k+1)Y(k+1)K(k+1)'$$

$$-K(k+1)\left(X(k+1)+\widetilde{H}(k+1)S(k+1|k)\right)-\left(X(k+1)'+S(k+1|k)\widetilde{H}(k+1)'\right)K(k+1)'$$

$$
\begin{aligned}
&= S(k+1|k) + K(k+1)\,Y(k+1)\,K(k+1)' \\
&\quad -K(k+1)\,Y(k+1)\,Y^{-1}(k+1)\left(X(k+1) + \widetilde{H}(k+1)\,S(k+1|k)\right) \\
&\quad -\left(X(k+1)' + S(k+1|k)\,\widetilde{H}(k+1)'\right)Y^{-1}(k+1)\,Y(k+1)\,K(k+1)' \\
&= S(k+1|k) + K(k+1)\,Y(k+1)\,K(k+1)' - K(k+1)\,Y(k+1)\,Z(k+1)' \\
&\quad -Z(k+1)\,Y(k+1)\,K(k+1)' \\
&= S(k+1|k) + K(k+1)\,Y(k+1)\,K(k+1)' - K(k+1)\,Y(k+1)\,Z(k+1)' \\
&\quad -Z(k+1)\,Y(k+1)\,K(k+1)' \\
&\quad +Z(k+1)\,Y(k+1)\,Z(k+1)' - Z(k+1)\,Y(k+1)\,Z(k+1)' \\
&= S(k+1|k) + \left(K(k+1) - Z(k+1)\right)Y(k+1)\left(K(k+1)' - Z(k+1)'\right) \\
&\quad -Z(k+1)\,Y(k+1)\,Z(k+1)' \;. \qquad (3\text{-}33)
\end{aligned}
$$

We see that (3-33) is minimized by setting $K(k+1) = Z(k+1)$. The optimal filter gain (i.e., optimal Kalman gain matrix) is

$$
\begin{aligned}
K(k+1) = Z(k+1) &= \left(S(k+1|k)\,\widetilde{H}(k+1)' + \Phi(k+1,k)\,D(k|k)\,\Lambda\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)'\right) \\
&\quad \times\left(\widetilde{H}(k+1)\,S(k+1|k)\,\widetilde{H}(k+1)' + \widetilde{R}(k+1)\right. \\
&\quad +\widetilde{H}(k+1)\,\Phi(k+1,k)\,D(k|k)\,\Lambda\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)' \\
&\quad \left.+\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)\Lambda D(k|k)'\,\Phi(k+1,k)'\,\widetilde{H}(k+1)'\right)^{-1} \;, \qquad (3\text{-}34)
\end{aligned}
$$

which is an n by q matrix. The form used below is

$$
\begin{aligned}
&K(k+1)\left(\widetilde{H}(k+1)\,S(k+1|k)\,\widetilde{H}(k+1)' + \widetilde{R}(k+1)\right. \\
&\quad +\widetilde{H}(k+1)\,\Phi(k+1,k)\,D(k|k)\,\Lambda\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)' \\
&\quad \left.+\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)\Lambda D(k|k)'\,\Phi(k+1,k)'\,\widetilde{H}(k+1)'\right) \\
&= S(k+1|k)\,\widetilde{H}(k+1)' + \Phi(k+1,k)\,D(k|k)\,\Lambda\left(W(k+1)\,\frac{\partial u}{\partial \lambda}\right)' \;. \qquad (3\text{-}35)
\end{aligned}
$$

Filter Development—Steady-State Case

We now examine the steady-state case of our problem. Referencing equations (41-43) of [7] we have

$$
\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \;,\; \Phi^{-1} = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix} \;, \qquad (3\text{-}36)
$$

where T is a step size in seconds,

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} , \tag{3-37}$$

and

$$W = 1 . \tag{3-38}$$

There are three cases for B. First, let

$$B_{\mathbf{a}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{3-39}$$

We have that $m(k)$ is a velocity noise in this case. In the other two cases, $m(k)$ is an acceleration noise. In these cases

$$B_{\mathbf{b}} = \begin{bmatrix} 0 \\ T \end{bmatrix} \tag{3-40}$$

and

$$B_{\mathbf{c}} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} . \tag{3-41}$$

In (3-39), process noise enters the system by the velocity state as velocity. In (3-40), process noise enters the system by the velocity state as acceleration multiplied by time. This noise affects the position state by way of the integration in the dynamics. This is similar to the set-up in Benedict and Bordner [3]. In (3-41), process noise enters the system as acceleration in the position state and velocity state. This arrangement is similar to the set-up in Kalata [6].

Case a: $B_a = \begin{bmatrix} 0 & 1 \end{bmatrix}'$

We treat the first case then, subsequently, appropriately modify various equations to adapt to the other two cases. Hence, considering B as in (3-39), that is $B = B_{\mathbf{a}}$, with p as position, v as velocity, and z as the measurement, the state transition and output equations for the steady-state case are

$$\begin{bmatrix} p(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m(k) \tag{3-42}$$

$$z(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p(k) \\ v(k) \end{bmatrix} + v(k) + u(x(k), \lambda(k)) . \tag{3-43}$$

Setting $u(x, \lambda) = \lambda$, the linear approximations from (3-9) become

$$\left. \frac{\partial u}{\partial x} \right|_{x=\widehat{x}(k|k), \lambda=\overline{\lambda}} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{3-44}$$

and

$$\left. \frac{\partial u}{\partial \lambda} \right|_{x=\widehat{x}(k|k), \lambda=\overline{\lambda}} = 1 . \tag{3-45}$$

Then, substituting (3-44) into (3-29)

$$\widetilde{H} = H + W \frac{\partial u}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} = H \tag{3-46}$$

and substituting (3-45) into (3-30)

$$\widetilde{R} = R + W\frac{\partial u}{\partial \lambda}\Lambda\frac{\partial u'}{\partial \lambda}W' = R + 1 \cdot 1 \cdot \Lambda \cdot 1 \cdot 1 = R + \Lambda \ . \tag{3-47}$$

Considering (3-15),

$$\widetilde{m} = \Phi^{-1}Bm = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} m = \begin{bmatrix} -T \\ 1 \end{bmatrix} m \ .$$

From (3-16) and (3-39) we have that

$$\widetilde{Q} \equiv \Phi^{-1}BQB'\left(\Phi^{-1}\right)' = \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} Q \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -T & 1 \end{bmatrix} = \begin{bmatrix} T^2 & -T \\ -T & 1 \end{bmatrix} Q \ . \tag{3-48}$$

The steady-state filter gain is

$$\overline{K} \equiv \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \ . \tag{3-49}$$

As mentioned, (3-49) is where α and β fit in for the Kalman gain matrix $K(k)$ given by (3-34). These gains are obtained by computing the steady-state values for all variables in (3-34). The objective of this section is to find a relationship between α and β.

The steady-state version of $L(k)$ from (3-7), $\overline{L}$, is

$$\overline{L}(k) = \left(I - \overline{K}H\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha & 0 \\ -\beta/T & 1 \end{bmatrix} \ . \tag{3-50}$$

The steady-state version of $F(k)$ from (3-13), $\overline{F}$, is

$$\overline{F} = \left(\begin{bmatrix} 1-\alpha & 0 \\ -\beta/T & 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\beta/T & 1-\beta \end{bmatrix} \ . \tag{3-51}$$

The eigenvalues of $\overline{F}$ are

$$e_{1,2} = 1 - \frac{(\alpha+\beta)}{2} \pm \frac{1}{2}\sqrt{2\alpha\beta - 4\beta + \alpha^2 + \beta^2} \ . \tag{3-52}$$

Then, referring to (3-14), the steady-state version of $C(k)$, $\overline{C}$, is

$$\overline{C} = -\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \cdot 1 \cdot 1 = -\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \ . \tag{3-53}$$

The measurement updated steady-state covariance $M(k|k)$, referring to (3-23), is

$$\overline{M} \equiv \lim_{k\to\infty} M(k|k) = \overline{F}\,\overline{M}\,\overline{F}' + \overline{F}\widetilde{Q}\overline{F}' + \overline{K}\,R\,\overline{K}' \ . \tag{3-54}$$

Using the superposition principle by letting

$$\overline{M} = \overline{M}_Q + \overline{M}_R \tag{3-55}$$

and then solving

$$\overline{M}_R = \overline{F}\,\overline{M}_R\,\overline{F}' + \overline{K}\,R\,\overline{K}' \tag{3-56}$$

$$\overline{M}_Q = \overline{F}\,\overline{M}_Q\,\overline{F}' + \overline{F}\widetilde{Q}\overline{F}' \ . \tag{3-57}$$

Comparing (3-49), (3-51), and (3-56) to (46), (47), and (48) of [7], we see that our solution for $\overline{M}_R$ is of the same form as (49) of [7]. Consequently,

$$\overline{M}_R = \frac{R}{\alpha\,(4-2\alpha-\beta)}\begin{bmatrix} 2\alpha^2+2\beta-3\alpha\beta & \beta\,(2\alpha-\beta)\,/T \\ \beta\,(2\alpha-\beta)\,/T & 2\beta^2/T^2 \end{bmatrix} . \tag{3-58}$$

The solution of $\overline{M}_Q$ remains to be determined. From (3-48) and (3-51),

$$\begin{aligned}\overline{F}\widetilde{Q}\overline{F}' &= \begin{bmatrix} 1-\alpha & (1-\alpha)\,T \\ -\beta/T & 1-\beta \end{bmatrix}\begin{bmatrix} T^2 & -T \\ -T & 1 \end{bmatrix}\cdot Q \cdot \begin{bmatrix} 1-\alpha & (1-\alpha)\,T \\ -\beta/T & 1-\beta \end{bmatrix}' \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\cdot Q \ . \end{aligned} \tag{3-59}$$

Substituting (3-51) and (3-59) into (3-57) gives

$$\overline{M}_Q = \begin{bmatrix} 1-\alpha & (1-\alpha)\,T \\ -\beta/T & 1-\beta \end{bmatrix}\overline{M}_Q\begin{bmatrix} 1-\alpha & -\beta/T \\ (1-\alpha)\,T & 1-\beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\cdot Q \ . \tag{3-60}$$

In longhand,

$$\begin{aligned}\overline{M}_Q &= \begin{bmatrix} \overline{m}_{11Q} & \overline{m}_{12Q} \\ \overline{m}_{12Q} & \overline{m}_{22Q} \end{bmatrix} \\ &= \left[\begin{matrix} (1-\alpha)^2\left(\overline{m}_{11Q}+2T\overline{m}_{12Q}+T^2\overline{m}_{22Q}\right) \\ (1-\alpha)\left(-\beta\overline{m}_{11Q}/T+(1-2\beta)\,\overline{m}_{12Q}+T\,(1-\beta)\,\overline{m}_{22Q}\right) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} (1-\alpha)\left(-\beta\overline{m}_{11Q}/T+(1-2\beta)\,\overline{m}_{12Q}+T\,(1-\beta)\,\overline{m}_{22Q}\right) \\ \left(\beta^2/T^2\right)\overline{m}_{11Q}+\left(2\beta\,(\beta-1)\,/T\right)\overline{m}_{12Q}+\left(1-2\beta+\beta^2\right)\overline{m}_{22Q} \end{matrix}\right] + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\cdot Q \ . \end{aligned} \tag{3-61}$$

Hence,

$$\begin{aligned}&\left[\begin{matrix} \overline{m}_{11Q}-(1-\alpha)^2\left(\overline{m}_{11Q}+2T\overline{m}_{12Q}+T^2\overline{m}_{22Q}\right) \\ \overline{m}_{12Q}-(1-\alpha)\left(-\beta\overline{m}_{11Q}/T+(1-2\beta)\,\overline{m}_{12Q}+T\,(1-\beta)\,\overline{m}_{22Q}\right) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} \overline{m}_{12Q}-(1-\alpha)\left(-\beta\overline{m}_{11Q}/T+(1-2\beta)\,\overline{m}_{12Q}+T\,(1-\beta)\,\overline{m}_{22Q}\right) \\ \overline{m}_{22Q}-\left(\beta^2/T^2\right)\overline{m}_{11Q}-\left(2\beta\,(\beta-1)\,/T\right)\overline{m}_{12Q}-\left(1-2\beta+\beta^2\right)\overline{m}_{22Q} \end{matrix}\right] \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\cdot Q \ . \end{aligned} \tag{3-62}$$

We get three equations in the three unknowns $\overline{m}_{11Q}$, $\overline{m}_{12Q}$, and $\overline{m}_{22Q}$ (defining A by the 3-by-3 matrix on the right hand side),

$$A\begin{bmatrix} \overline{m}_{11Q} \\ \overline{m}_{12Q} \\ \overline{m}_{22Q} \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)^2 & -2\,(1-\alpha)^2\,T & -(1-\alpha)^2\,T^2 \\ (1-\alpha)\,\beta/T & 1-(1-\alpha)\,(1-2\beta) & -(1-\alpha)\,(1-\beta)\,T \\ -\left(\beta^2/T^2\right) & -2\beta\,(\beta-1)\,/T & 1-\left(1-2\beta+\beta^2\right) \end{bmatrix}\begin{bmatrix} \overline{m}_{11Q} \\ \overline{m}_{12Q} \\ \overline{m}_{22Q} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix} . \tag{3-63}$$

where (3-63) also defines the 3-by-3 matrix A. Taking the matrix inverse of A to solve for $\overline{M}_Q$,

$$\begin{bmatrix} \overline{m}_{11Q} \\ \overline{m}_{12Q} \\ \overline{m}_{22Q} \end{bmatrix} = \begin{bmatrix} 1-(1-\alpha)^2 & -2(1-\alpha)^2 T & -(1-\alpha)^2 T^2 \\ (1-\alpha)\beta/T & 1-(1-\alpha)(1-2\beta) & -(1-\alpha)(1-\beta)T \\ -\left(\beta^2/T^2\right) & -2\beta(\beta-1)/T & 1-\left(1-2\beta+\beta^2\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ Q \end{bmatrix} .$$

The determinant of A is: $\det(A) = 4\alpha\beta - \alpha\beta^2 - 2\alpha^2\beta = \alpha\beta(4-\beta-2\alpha)$; and should not be zero for the inverse to exist. This is satisfied by these conditions:

$$\begin{array}{ll} 1. & \alpha \neq 0 \\ 2. & \beta \neq 0 \\ 3. & \beta \neq 4-2\alpha \end{array} . \tag{3-64}$$

If the determinant of A is not zero, we can obtain the solution:

$$\begin{bmatrix} \overline{m}_{11Q} \\ \overline{m}_{12Q} \\ \overline{m}_{22Q} \end{bmatrix} = Q \cdot \begin{bmatrix} T^2\left(-2+5\alpha-4\alpha^2+\alpha^3\right) \\ T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) \\ \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix} / \left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right) . \tag{3-65}$$

In matrix form,

$$\overline{M}_Q = \frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} \begin{bmatrix} T^2\left(-2+5\alpha-4\alpha^2+\alpha^3\right) & T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) \\ T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix} . \tag{3-66}$$

And finally $\overline{M}$ is obtained from (3-55), (3-58) and (3-66):

$$\overline{M} = \frac{R}{\alpha(4-2\alpha-\beta)} \begin{bmatrix} 2\alpha^2+2\beta-3\alpha\beta & \beta(2\alpha-\beta)/T \\ \beta(2\alpha-\beta)/T & 2\beta^2/T^2 \end{bmatrix}$$

$$+\frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} \begin{bmatrix} T^2\left(-2+5\alpha-4\alpha^2+\alpha^3\right) & T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) \\ T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix} . \tag{3-67}$$

We see that $\overline{m}_{11} = \overline{m}_{11}(\alpha,\beta,T,R,Q)$, $\overline{m}_{12} = \overline{m}_{12}(\alpha,\beta,T,R,Q)$ and $\overline{m}_{22} = \overline{m}_{22}(\alpha,\beta,T,R,Q)$. The usual technique for solving the Liapunov equation (3-54) is by algebraic manipulation and using the symmetry of the matrix, as demonstrated with the solution (3-67). Numerical solutions may be obtained by repeated propagation until arriving at steady-state. We note that $\overline{m}_{11}/Q$ is the so-called *noise reduction factor*; see [2].

The time-updated steady-state covariance $\overset{\circ}{M}$, referring to (3-22), is

$$\overset{\circ}{M} \equiv \lim_{k\to\infty} M(k+1|k) = \lim_{k\to\infty} \Phi M(k|k)\,\Phi' + BQB' = \Phi\left(\overline{M}_R + \overline{M}_Q\right)\Phi' + BQB' . \tag{3-68}$$

We get

$$\Phi\overline{M}_R\Phi' = \frac{R}{\alpha\left(4-2\alpha-\beta\right)}\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2\alpha^2+2\beta-3\alpha\beta & \beta\left(2\alpha-\beta\right)/T \\ \beta\left(2\alpha-\beta\right)/T & 2\beta^2/T^2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix}$$

$$= \frac{R}{\alpha\left(4-2\alpha-\beta\right)}\begin{bmatrix} 2\alpha^2+2\beta+\alpha\beta & \beta\left(2\alpha+\beta\right)/T \\ \beta\left(2\alpha+\beta\right)/T & 2\beta^2/T^2 \end{bmatrix}$$

and

$$\Phi\overline{M}_Q\Phi' =$$

$$= \frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} T^2\left(-2+5\alpha-4\alpha^2+\alpha^3\right) & T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) \\ T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix}$$

$$= \frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\begin{bmatrix} T^2\left(-2+\alpha\right) & T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) \\ T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix} .$$

Hence,

$$\mathring{M} = \frac{R}{\alpha\left(4-2\alpha-\beta\right)}\begin{bmatrix} 2\alpha^2+2\beta+\alpha\beta & \beta\left(2\alpha+\beta\right)/T \\ \beta\left(2\alpha+\beta\right)/T & 2\beta^2/T^2 \end{bmatrix}$$
$$+\frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\begin{bmatrix} T^2\left(-2+\alpha\right) & T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) \\ T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} . \tag{3-69}$$

We need steady-state versions of these: designating

$$\overline{D} \equiv \lim_{k\to\infty} D\left(k|k\right)$$

and

$$\mathring{D} \equiv \lim_{k\to\infty} D\left(k+1|k\right) \ ,$$

we then have, referring to (3-25) and (3-27),

$$\mathring{D} = \overline{F}\,\overline{D}$$

$$\overline{D} = \mathring{D} + \overline{C}$$

then

$$\mathring{D} = \overline{D} - \overline{C} \ .$$

Continuing,

$$\overline{F}\,\overline{D} = \overline{D} - \overline{C}$$

$$\overline{F}\,\overline{D} - \overline{D} = \left(\overline{F} - I\right)\overline{D} = -\overline{C} \ ;$$

hence,

$$\overline{D} = -\left(\overline{F} - I\right)^{-1}\overline{C} \ .$$

Using (3-51), and (3-53)

$$\overline{D} = \left(\begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\beta/T & 1-\beta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\alpha & (1-\alpha)T \\ -\beta/T & -\beta \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -\beta & -(1-\alpha)T \\ \beta/T & -\alpha \end{bmatrix}}{\beta} \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} .$$

Hence,

$$\overset{\circ}{D} = \overline{F}\,\overline{D} = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\beta/T & 1-\beta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha-1 \\ \beta/T \end{bmatrix} .$$

Finally, the steady-state time-updated total (due to noise and bias) covariance, $\overset{\circ}{S}$, is obtained by substituting into (3-28),

$$\overset{\circ}{S} = \begin{bmatrix} \overset{\circ}{S}_{11} & \overset{\circ}{S}_{12} \\ \overset{\circ}{S}_{21} & \overset{\circ}{S}_{22} \end{bmatrix} = \overset{\circ}{M} + \Phi\overline{D}\Lambda\overline{D}'\Phi'$$

$$= \begin{bmatrix} \overset{\circ}{M}_{11} & \overset{\circ}{M}_{12} \\ \overset{\circ}{M}_{21} & \overset{\circ}{M}_{22} \end{bmatrix} + \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \Lambda \cdot \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{\circ}{M}_{11} & \overset{\circ}{M}_{12} \\ \overset{\circ}{M}_{21} & \overset{\circ}{M}_{22} \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} . \tag{3-70}$$

So, using (3-69),

$$\begin{bmatrix} \overset{\circ}{S}_{11} & \overset{\circ}{S}_{12} \\ \overset{\circ}{S}_{21} & \overset{\circ}{S}_{22} \end{bmatrix} = \frac{R}{\alpha(4-2\alpha-\beta)} \begin{bmatrix} 2\alpha^2+2\beta+\alpha\beta & \beta(2\alpha+\beta)/T \\ \beta(2\alpha+\beta)/T & 2\beta^2/T^2 \end{bmatrix}$$

$$+\frac{Q}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} \begin{bmatrix} T^2(-2+\alpha) & T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) \\ T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} .$$

In particular,

$$\overset{\circ}{S}_{11} = \overset{\circ}{M}_{11} + \Lambda$$

$$= \frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha(4-2\alpha-\beta)} + \frac{QT^2(-2+\alpha)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + \Lambda \tag{3-71}$$

and

$$\overset{\circ}{S}_{21} = \overset{\circ}{M}_{21}$$

$$= \frac{R\beta\,(2\alpha+\beta)}{\alpha\,(4-2\alpha-\beta)\,T} + \frac{QT\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\ . \tag{3-72}$$

We now turn our attention to (3-35). We first simplify the summands in the left hand side of (3-35). In steady-state,

$$\widetilde{H}\overset{\circ}{S}\widetilde{H}' + \widetilde{R} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \overset{\circ}{S}_{11} & \overset{\circ}{S}_{12} \\ \overset{\circ}{S}_{21} & \overset{\circ}{S}_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + R + \Lambda$$

$$= \overset{\circ}{S}_{11} + R + \Lambda\ . \tag{3-73}$$

Also,

$$\widetilde{H}\Phi\overline{D}\Lambda\left(W\frac{\partial u}{\partial\lambda}\right)' = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \Lambda \cdot 1 \cdot 1 = -\Lambda\ , \tag{3-74}$$

and

$$\left(W\frac{\partial u}{\partial\lambda}\right)\Lambda\overline{D}'\Phi'\widetilde{H}' = -\Lambda\ . \tag{3-75}$$

For the right hand side of (3-35) in steady-state,

$$\overset{\circ}{S}\widetilde{H}' = \begin{bmatrix} \overset{\circ}{S}_{11} & \overset{\circ}{S}_{12} \\ \overset{\circ}{S}_{12} & \overset{\circ}{S}_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \overset{\circ}{S}_{11} \\ \overset{\circ}{S}_{12} \end{bmatrix}\ , \tag{3-76}$$

and

$$\Phi\overline{D}\Lambda\left(W\frac{\partial u}{\partial\lambda}\right)' = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \cdot \Lambda \cdot 1 \cdot 1 = \begin{bmatrix} -\Lambda \\ 0 \end{bmatrix}\ . \tag{3-77}$$

Substituting (3-73), through (3-77) into (3-35) gives

$$\begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \left(\overset{\circ}{S}_{11} + R + \Lambda - \Lambda - \Lambda\right) = \begin{bmatrix} \overset{\circ}{S}_{11} \\ \overset{\circ}{S}_{12} \end{bmatrix} + \begin{bmatrix} -\Lambda \\ 0 \end{bmatrix}\ .$$

This, written as two scalar equations,

$$\alpha\left(\overset{\circ}{S}_{11} + R + \Lambda - \Lambda - \Lambda\right) = \alpha\left(\overset{\circ}{S}_{11} + R - \Lambda\right) = \overset{\circ}{S}_{11} - \Lambda \tag{3-78}$$

$$\frac{\beta}{T}\left(\overset{\circ}{S}_{11} + R + \Lambda - \Lambda - \Lambda\right) = \frac{\beta}{T}\left(\overset{\circ}{S}_{11} + R - \Lambda\right) = \overset{\circ}{S}_{12}\ . \tag{3-79}$$

Substituting (3-71) and (3-72)

$$\alpha\left(\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\,(4-2\alpha-\beta)} + \frac{QT^2\,(-2+\alpha)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + \Lambda\right) + R - \Lambda\right)$$

$$= \frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\,(4-2\alpha-\beta)} + \frac{QT^2\,(-2+\alpha)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + \Lambda - \Lambda$$

$$\beta\left(\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}+\Lambda\right)+R-\Lambda\right)$$
$$=\left(\frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)T}+\frac{QT\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\right)\cdot T\ .$$

We will later remark on the fact that Λ cancels out at this point:

$$\alpha\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}+R\right)$$
$$=\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}$$
$$\beta\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}+R\right)$$
$$=\frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\ .$$

We define $\rho^2 = QT^2/R$. With Q in $(m/\sec)^2$, T in seconds and R in m^2, ρ is unitless. Substituting this in the previous two equations, we obtain

$$\alpha\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}+1\right)$$
$$=\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}$$
$$\beta\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}+1\right)$$
$$=\frac{\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}\ .$$

We divide the previous two equations and cancel an α,

$$\frac{\alpha}{\beta}=\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)\left(-4\beta+\beta^2+2\alpha\beta\right)+\rho^2\left(-2+\alpha\right)\left(4-2\alpha-\beta\right)}{\beta\left(2\alpha+\beta\right)\left(-4\beta+\beta^2+2\alpha\beta\right)+\rho^2\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)\left(4-2\alpha-\beta\right)}\ .$$

Cross multiplying gives:

$$2\beta^4+\left(4\alpha-8\right)\beta^3+\rho^2\left(\alpha^2-2\alpha+2\right)\beta^2+$$
$$\rho^2\left(3\alpha^3-10\alpha^2+12\alpha-8\right)\beta+\rho^2\left(2\alpha^4-8\alpha^3+8\alpha^2\right)=0\ . \qquad (3\text{-}80)$$

Equation (3-80) gives our relationship between α and β. The noise ratio ρ, a parameter in the equation, is known, or known to be within a range. Equation (3-80) may be factored[6]

$$(\beta + (2\alpha - 4))\left(2\beta^3 + \rho^2\left(\left(\alpha^2 - 2\alpha + 2\right)\beta + \alpha^2(\alpha - 2)\right)\right) = 0\ . \tag{3-81}$$

Hence, $\beta = (4 - 2\alpha)$ is a solution that is independent of ρ. This solution is not permitted since it violates Condition 3 of (3-64). There is a second real solution for β given α (which depends on ρ.) The remaining two solutions for β given α may be a complex conjugate pair. The table below gives representative solutions to (3-80). The two real solutions are presented. We do not use the second one listed because of Condition 3. Appendix B gives the solution to the cubic equation part of (3-81). Of course, the Newton-Raphson method may be used to compute all of the solutions to (3-80).

ρ^2	α	β
2	0.2	0.04385, 3.6
4	0.2	0.04386, 3.6
6	0.2	0.04389, 3.6
6	0.4	0.1866, 3.2
8	0.2	0.04389, 3.6
8	0.4	0.1870, 3.2
10	0.2	0.04389, 3.6
10	0.4	0.1873, 3.2
10	0.5	0.2959, 3.0

These α and β give that the eigenvalues of $\overline{F}$, in (3-52), have norm less than 1. Hence, by Theorem 2.1, page 64 of Anderson and Moore [1], M_R and M_Q, the solutions to (3-56) and (3-57) respectively, exist, are unique, and are positive definite.

Case b: $B_b = \left[\begin{array}{cc} 0 & T \end{array}\right]'$

We consider the development of this section using (3-40), that is $B = B_{\mathbf{b}}$. The effect of (3-40) on (3-15) is

$$\widetilde{m} = \Phi^{-1}Bm = \left[\begin{array}{cc} 1 & -T \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} 0 \\ T \end{array}\right] m = \left[\begin{array}{c} -T^2 \\ T \end{array}\right] m$$

Continuing, we observe that, roughly speaking, we replace every "Q" in the above development with a "QT^2". Specifically, (3-48) becomes

$$\widetilde{Q} \equiv \Phi^{-1}BQB'\Phi^{-1\prime} = \left[\begin{array}{cc} T^2 & -T \\ -T & 1 \end{array}\right] Q \cdot T^2\ . \tag{3-82}$$

Also, (3-59) becomes

$$\overline{F}\widetilde{Q}\overline{F}' = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right] Q \cdot T^2\ . \tag{3-83}$$

In the development involving $\overline{M}_Q$, (3-60) through (3-66), we multiply every Q by a T^2. Doing this, (3-66) becomes

[6] Provided by Armido R. DiDonato.

$$\overline{M}_Q = \frac{Q \cdot T^2}{\left(-4\alpha\beta + \alpha\beta^2 + 2\alpha^2\beta\right)} \left[\begin{array}{cc} T^2\left(-2+5\alpha-4\alpha^2+\alpha^3\right) & T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) \\ T\left(-2\alpha+\beta-\alpha\beta+3\alpha^2-\alpha^3\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{array} \right] . \quad (3\text{-}84)$$

These changes continue on to $\overline{M}$ and $\mathring{M}$, giving that (3-69) becomes

$$\mathring{M} = \frac{R}{\alpha\left(4-2\alpha-\beta\right)} \left[\begin{array}{cc} 2\alpha^2+2\beta+\alpha\beta & \beta\left(2\alpha+\beta\right)/T \\ \beta\left(2\alpha+\beta\right)/T & 2\beta^2/T^2 \end{array} \right]$$

$$+\frac{Q \cdot T^2}{\left(-4\alpha\beta + \alpha\beta^2 + 2\alpha^2\beta\right)} \left[\begin{array}{cc} T^2\left(-2+\alpha\right) & T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) \\ T\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right) & \left(-2\beta+2\alpha\beta-2\alpha^2+\alpha^3\right) \end{array} \right]$$

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \cdot T^2 Q .$$

Finally, looking at the effect of (3-40) on the steady-state time-update total covariance, putting in T^2Q for Q in (3-71) we obtain

$$\begin{aligned} \mathring{S}_{11} &= \mathring{M}_{11} + \Lambda \\ &= \frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)} + \frac{QT^4\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + \Lambda \end{aligned} \quad (3\text{-}85)$$

and doing the same in (3-72) gives

$$\begin{aligned} \mathring{S}_{21} &= \mathring{M}_{21} \\ &= \frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)T} + \frac{QT^3\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} . \end{aligned} \quad (3\text{-}86)$$

Substitution of (3-85) and (3-86) into (3-78) and (3-79) gives

$$\alpha\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)} + \frac{QT^4\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + R\right)$$

$$= \frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)} + \frac{QT^4\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}$$

$$\beta\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)} + \frac{QT^4\left(-2+\alpha\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} + R\right)$$

$$= \frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)} + \frac{QT^4\left(-2\alpha-\beta+\alpha\beta+\alpha^2\right)}{\left(-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)} .$$

This time we define $\rho^2 = QT^4/R$. With Q in $\left(m/\sec^2\right)^2$, T in seconds, and R in m^2, ρ is again unitless. We again obtain the result (3-81), but with ρ^2 interpreted differently.

Case c: $B_c = \left[\begin{array}{cc} T^2/2 & T \end{array} \right]'$

We consider the development of this section using (3-41), that is $B = B_{\mathbf{c}}$. The changes are more extensive than they were before with (3-40). The effect of (3-41) on (3-15) is

$$\widetilde{m} = \Phi^{-1} B m = \left[\begin{array}{cc} 1 & -T \\ 0 & 1 \end{array} \right] \left[\begin{array}{c} T^2/2 \\ T \end{array} \right] m = \left[\begin{array}{c} -T^2/2 \\ T \end{array} \right] m$$

For (3-48),

$$\widetilde{Q} \equiv \Phi^{-1} B Q B' \Phi^{-1\prime} = \left[\begin{array}{c} -T^2/2 \\ T \end{array} \right] Q \left[\begin{array}{cc} -T^2/2 & T \end{array} \right] = \left[\begin{array}{cc} T^2/4 & -T/2 \\ -T/2 & 1 \end{array} \right] Q \cdot T^2 . \tag{3-87}$$

Continuing, we find

$$\overline{F} \widetilde{Q} \overline{F}' = \left[\begin{array}{cc} 1-\alpha & (1-\alpha) T \\ -\beta/T & 1-\beta \end{array} \right] \left[\begin{array}{cc} T^2/4 & -T/2 \\ -T/2 & 1 \end{array} \right] Q \cdot T^2 \left[\begin{array}{cc} 1-\alpha & (1-\alpha) T \\ -\beta/T & 1-\beta \end{array} \right]'$$

$$= \left[\begin{array}{cc} T^2 (1-\alpha)^2 & T (1-\alpha)(2-\beta) \\ T (1-\alpha)(2-\beta) & (\beta-2)^2 \end{array} \right] \cdot \frac{QT^2}{4} ,$$

and this is considerably different from (3-59). As before, (3-63),

$$\left[\begin{array}{c} \overline{m}_{11Q} \\ \overline{m}_{12Q} \\ \overline{m}_{22Q} \end{array} \right] = \left[\begin{array}{ccc} 1-(1-\alpha)^2 & -2(1-\alpha)^2 T & -(1-\alpha)^2 T^2 \\ (1-\alpha)\beta/T & 1-(1-\alpha)(1-2\beta) & -(1-\alpha)(1-\beta) T \\ -\left(\beta^2/T^2\right) & -2\beta(\beta-1)/T & 1-\left(1-2\beta+\beta^2\right) \end{array} \right]^{-1}$$

$$\cdot \left[\begin{array}{c} T^2 (1-\alpha)^2 \\ T(1-\alpha)(2-\beta) \\ (\beta-2)^2 \end{array} \right] \cdot \frac{QT^2}{4} .$$

As before, the matrix that is inverted is referred to as "A", and we require the same conditions on its determinant, which are given in (3-64). We find that

$$\overline{M}_Q = \frac{QT^2}{4\left(4\alpha\beta - \alpha\beta^2 - 2\alpha^2\beta\right)} \left[\begin{array}{cc} T^2\left(8-20\alpha-2\beta+4\alpha\beta+16\alpha^2-4\alpha^3-2\alpha^2\beta\right) & T\left(8\alpha-4\beta+4\alpha\beta-12\alpha^2+4\alpha^3+\beta^2-\alpha\beta^2\right) \\ T\left(8\alpha-4\beta+4\alpha\beta-12\alpha^2+4\alpha^3+\beta^2-\alpha\beta^2\right) & 8\beta-16\alpha\beta+8\alpha^2-4\alpha^3-2\beta^2+3\alpha\beta^2+4\alpha^2\beta \end{array} \right] . \tag{3-88}$$

In this case, we obtain $\overline{M}$ from (3-55), (3-58) and (3-88):

$$\overline{M} = \frac{R}{\alpha(4-2\alpha-\beta)} \left[\begin{array}{cc} 2\alpha^2+2\beta-3\alpha\beta & \beta(2\alpha-\beta)/T \\ \beta(2\alpha-\beta)/T & 2\beta^2/T^2 \end{array} \right]$$

$$+ \frac{QT^2}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)} \left[\begin{array}{cc} 8T^2-20T^2\alpha-2T^2\beta+4T^2\alpha\beta+16T^2\alpha^2-4T^2\alpha^3-2T^2\alpha^2\beta & 8T\alpha-4T\beta+4T\alpha\beta-12T\alpha^2+4T\alpha^3+T\beta^2-T\alpha\beta^2 \\ 8T\alpha-4T\beta+4T\alpha\beta-12T\alpha^2+4T\alpha^3+T\beta^2-T\alpha\beta^2 & 8\beta-16\alpha\beta+8\alpha^2-4\alpha^3-2\beta^2+3\alpha\beta^2+4\alpha^2\beta \end{array} \right] . \tag{3-89}$$

Moving on to the time-updated steady-state covariance, we first compute

$$\Phi\overline{M}_Q\Phi' = \frac{QT^2}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}\left[\begin{array}{cc} 8T^2-4T^2\alpha-2T^2\beta-4T^2\alpha\beta+T^2\alpha\beta^2+2T^2\alpha^2\beta & 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta \\ 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta & 8\beta-16\alpha\beta+8\alpha^2-4\alpha^3-2\beta^2+3\alpha\beta^2+4\alpha^2\beta \end{array}\right].$$

and note that $\Phi\overline{M}_R\Phi'$ is the same as before. Following (3-68) we obtain

$$\overset{\circ}{M} = \frac{R}{\alpha\left(4-2\alpha-\beta\right)}\left[\begin{array}{cc} 2\alpha^2+2\beta+\alpha\beta & \beta\left(2\alpha+\beta\right)/T \\ \beta\left(2\alpha+\beta\right)/T & 2\beta^2/T^2 \end{array}\right]$$

$$+\frac{Q\cdot T^2}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}\left[\begin{array}{cc} 8T^2-4T^2\alpha-2T^2\beta-4T^2\alpha\beta+T^2\alpha\beta^2+2T^2\alpha^2\beta & 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta \\ 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta & 8\beta-16\alpha\beta+8\alpha^2-4\alpha^3-2\beta^2+3\alpha\beta^2+4\alpha^2\beta \end{array}\right]+\left[\begin{array}{cc} \frac{1}{4}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{array}\right]Q\,. \tag{3-90}$$

We repeat (3-70)

$$\overset{\circ}{S} = \left[\begin{array}{cc} \overset{\circ}{M}_{11} & \overset{\circ}{M}_{12} \\ \overset{\circ}{M}_{21} & \overset{\circ}{M}_{22} \end{array}\right]+\left[\begin{array}{cc} \Lambda & 0 \\ 0 & 0 \end{array}\right]$$

and then substitute from (3-90)

$$\left[\begin{array}{cc} \overset{\circ}{S}_{11} & \overset{\circ}{S}_{12} \\ \overset{\circ}{S}_{21} & \overset{\circ}{S}_{22} \end{array}\right] = \frac{R}{\alpha\left(4-2\alpha-\beta\right)}\left[\begin{array}{cc} 2\alpha^2+2\beta+\alpha\beta & \beta\left(2\alpha+\beta\right)/T \\ \beta\left(2\alpha+\beta\right)/T & 2\beta^2/T^2 \end{array}\right]$$

$$+\frac{Q\cdot T^2}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}\left[\begin{array}{cc} 8T^2-4T^2\alpha-2T^2\beta-4T^2\alpha\beta+T^2\alpha\beta^2+2T^2\alpha^2\beta & 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta \\ 8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta & 8\beta-16\alpha\beta+8\alpha^2-4\alpha^3-2\beta^2+3\alpha\beta^2+4\alpha^2\beta \end{array}\right]$$

$$+\left[\begin{array}{cc} \frac{1}{4}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T^2 \end{array}\right]Q+\left[\begin{array}{cc} \Lambda & 0 \\ 0 & 0 \end{array}\right].$$

In particular (as before)

$$\overset{\circ}{S}_{11} = \overset{\circ}{M}_{11}+\Lambda$$

$$= \frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^2\left(8T^2-4T^2\alpha-2T^2\beta-4T^2\alpha\beta+T^2\alpha\beta^2+2T^2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}T^4Q+\Lambda \tag{3-91}$$

and

$$\overset{\circ}{S}_{21} = \overset{\circ}{M}_{21}$$

$$= \frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)T}+\frac{QT^2\left(8T\alpha+4T\beta-12T\alpha\beta-4T\alpha^2-T\beta^2+2T\alpha\beta^2+4T\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{T^3Q}{2}\,. \tag{3-92}$$

Substituting (3-91) and (3-92) into (3-78) and (3-79), we obtain

$$\alpha\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^4\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}T^4Q+R\right)$$

$$=\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^4\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}T^4Q\right)$$

and

$$\beta\left(\frac{R\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{QT^4\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}T^4Q+R\right)$$

$$=\left(\frac{R\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)T}+\frac{QT^3\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{T^3Q}{2}\right)\cdot T\ .$$

Again we define $\rho^2 = QT^4/R$. With Q in $\left(m/\sec^2\right)^2$, T in seconds, and R in m^2, ρ is again unitless. In this case, ρ is referred to as either the *target maneuver index* or the *target tracking index*; see [2]. Again we note that Λ drops out. This modifies the above as follows:

$$\alpha\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}\rho^2+1\right)$$

$$=\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}\rho^2\right)$$

and

$$\beta\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{4}\rho^2+1\right)$$

$$=\left(\frac{\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)}{4\left(4\alpha\beta-\alpha\beta^2-2\alpha^2\beta\right)}+\frac{1}{2}\rho^2\right)\ .$$

We divide the previous two equations,

$$\frac{\alpha}{\beta}=\frac{\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\alpha\beta\left(4-\beta-2\alpha\right)}+\frac{1}{4}\rho^2\right)}{\left(\frac{\beta\left(2\alpha+\beta\right)}{\alpha\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)}{4\alpha\beta\left(4-\beta-2\alpha\right)}+\frac{1}{2}\rho^2\right)}$$

$$=\frac{\left(\frac{\left(2\alpha^2+2\beta+\alpha\beta\right)}{\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{4\beta\left(4-\beta-2\alpha\right)}+\frac{1}{4}\rho^2\alpha\right)}{\left(\frac{\beta\left(2\alpha+\beta\right)}{\left(4-2\alpha-\beta\right)}+\frac{\rho^2\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)}{4\beta\left(4-\beta-2\alpha\right)}+\frac{1}{2}\rho^2\alpha\right)}$$

$$= \frac{\left(\frac{4\beta\left(2\alpha^2+2\beta+\alpha\beta\right)}{(4-2\alpha-\beta)} + \frac{\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)}{(4-\beta-2\alpha)} + \rho^2\alpha\beta\right)}{\left(\frac{4\beta^2\left(2\alpha+\beta\right)}{(4-2\alpha-\beta)} + \frac{\rho^2\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)}{(4-\beta-2\alpha)} + 2\rho^2\alpha\beta\right)}$$

$$= \frac{4\beta\left(2\alpha^2+2\beta+\alpha\beta\right)+\rho^2\left(8-4\alpha-2\beta-4\alpha\beta+\alpha\beta^2+2\alpha^2\beta\right)+\rho^2\alpha\beta\left(4-2\alpha-\beta\right)}{4\beta^2\left(2\alpha+\beta\right)+\rho^2\left(8\alpha+4\beta-12\alpha\beta-4\alpha^2-\beta^2+2\alpha\beta^2+4\alpha^2\beta\right)+2\rho^2\alpha\beta\left(4-2\alpha-\beta\right)}$$

$$= \frac{4\beta\left(2\alpha^2+2\beta+\alpha\beta\right)+2\rho^2\left(4-\beta-2\alpha\right)}{4\beta^2\left(2\alpha+\beta\right)+\rho^2\left(8\alpha+4\beta-4\alpha\beta-4\alpha^2-\beta^2\right)} \ .$$

Therefore,

$$\begin{aligned} &\alpha\left(4\beta^2\left(2\alpha+\beta\right)+\rho^2\left(8\alpha+4\beta-4\alpha\beta-4\alpha^2-\beta^2\right)\right) \\ &= \beta\left(4\beta\left(2\alpha^2+2\beta+\alpha\beta\right)+2\rho^2\left(4-\beta-2\alpha\right)\right) \ . \end{aligned}$$

Expanding,

$$\rho^2\left(8\alpha\beta-8\beta+8\alpha^2-4\alpha^3+2\beta^2-\alpha\beta^2-4\alpha^2\beta\right)-8\beta^3=0 \ . \tag{3-93}$$

Before proceeding with a numerical solution and demonstration of this equation, we will need to review the Kalata relationship [6]. With ρ a given, there are two equations in the two unknowns α and β:

$$\beta = (4-2\alpha)-4\sqrt{1-\alpha} \tag{3-94}$$

and

$$(4-2\alpha)-(4+\rho)\sqrt{1-\alpha}=0 \ . \tag{3-95}$$

Using (3-94), we can rewrite (3-95) as

$$\rho^2 = \frac{\beta^2}{(1-\alpha)} \ . \tag{3-96}$$

Equation (3-94) gives the Kalata relationship between α and β. When we know the correct ρ, (3-96) gives the applicable α and β. Equations (3-94) and (3-96) are in [6]; see page 176.

β from (3-93) given α and ρ		
ρ^2	α	β
2	0.2	0.044390
2	0.4	0.19393
4	0.2	0.044417
4	0.4	0.19683
6	0.2	0.044426
6	0.4	0.19785
8	0.2	0.044431
8	0.4	0.19837
10	0.2	0.044433
10	0.4	0.19869
10	0.5	0.32640

Optimal α and β given ρ (3-93)			$\alpha - \beta$ with Kalata relation	
α	β	ρ	α	β
0.36	0.08	0.1	0.36	0.08
0.50514	0.17587	0.25	0.50514	0.17587
0.62837	0.30481	0.5	0.62837	0.30481
0.75	0.5	1	0.75	0.5
0.85410	0.76393	2	0.85410	0.76393
0.92820	1.0718	4	0.92820	1.0718
0.97871	1.4590	10	0.97871	1.4590

As ρ approaches infinity, α approaches one and β approaches two. From this table, we see that (3-93) matches the Kalata relationships (3-95) and (3-96). Mookerjee and Reifler [7] also got the Kalata relationship between α and β for their problem as well. Not surprising, since as mentioned above, these are dual problems.

4 SUMMARY AND CONCLUSIONS

In this report we considered topics related to determining radar sensor bias. We presented an algorithm that estimates the absolute bias of two sensors when the relative bias between the sensors is given. The algorithm uses the relative bias, which is given in rectangular coordinates, as a constraint. The absolute biases, in spherical coordinates, for the sensors are obtained by the solution to an optimization problem that exploits the spherical-to-rectangular coordinate conversion. We presented a reduced-state filter designed for performance with sensor bias. The filter is reduced-state since it does not contain additional bias states. The filter design is influenced by the filter in Mookerjee and Reifler [7] and may be viewed as a dual design, in the control theory sense, to their filter [7].

A flow diagram for processing radar data with bias may contain these stages:

1. Estimate state with the $\alpha - \beta$ filter optimized for measurement bias, as presented in Chapter 3.

2. For a multi-sensor problem, estimate the relative sensor bias using an optimized algorithm such as in Brown, Weisman, and Brock [5].

3. Continue by estimating the absolute bias for each sensor using the algorithm presented in Chapter 2.

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APPENDIX A:

TRANSFORMATION FROM ENU(1) TO ENU(2)

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In this appendix, we present the transformation from the ENU(1), East North Up, to the ENU(2) coordinate systems. But first, consider the transformation from the Earth Centered Inertial, denoted ECI, to ENU. Consider an ENU coordinate axis located at longitude-latitude $\Omega - L$ and define the rotation matrix

$$T_{ECI2ENU} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos L & -\sin L \\ 0 & \sin L & \cos L \end{bmatrix} \begin{bmatrix} \cos\Omega & 0 & -\sin\Omega \\ 0 & 1 & 0 \\ \sin\Omega & 0 & \cos\Omega \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\Omega & \cos\Omega & 0 \\ -\sin L\cos\Omega & -\sin L\sin\Omega & \cos L \\ \cos L\cos\Omega & \cos L\sin\Omega & \sin L \end{bmatrix}$$

with $T_{ENU2ECI} = T'_{ECI2ENU}$. For position, we need to include a translation, so that for a given position vector in ENU coordinates

$$P_{ECI} = \frac{r_{ee}}{\sqrt{1 - e^2\sin^2 L}} \begin{bmatrix} \cos L\cos\Omega \\ \cos L\sin\Omega \\ \left(1 - e^2\right)\sin L \end{bmatrix} + T_{ENU2ECI} P_{ENU}$$

where r_{ee} is the earth's equatorial radius and e is the earth's eccentricity. For velocity, use the rotation alone.

Let the ENU(1), ENU(2) coordinate system be located at longitude-latitude $\{\Omega_1, L_1\}$ and $\{\Omega_2, L_2\}$ respectively. Next we consider our transformation going from $\{\Omega_1, L_1\}$ to $\{\Omega_2, L_2\}$. The rotation part of this transformation can be represented by the matrix below (going from ENU(1) to ENU(2)). There are three steps. First, the ENU(1) coordinates are rotated down to the equator. Second, these coordinates are rotated along the equator by the longitude difference. Third is the rotation up to the latitude of the ENU(2) system.

$$T_{ENU(1)2ENU(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos L_2 & -\sin L_2 \\ 0 & \sin L_2 & \cos L_2 \end{bmatrix} \begin{bmatrix} \cos(\Omega_2 - \Omega_1) & 0 & -\sin(\Omega_2 - \Omega_1) \\ 0 & 1 & 0 \\ \sin(\Omega_2 - \Omega_1) & 0 & \cos(\Omega_2 - \Omega_1) \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos L_1 & \sin L_1 \\ 0 & -\sin L_1 & \cos L_1 \end{bmatrix}$$

$$= \left[\begin{matrix} \cos(\Omega_2 - \Omega_1) & \sin L_1\sin(\Omega_2 - \Omega_1) \\ -\sin L_2\sin(\Omega_2 - \Omega_1) & \cos L_1\cos L_2 + \sin L_1\sin L_2\cos(\Omega_2 - \Omega_1) \\ \cos L_2\sin(\Omega_2 - \Omega_1) & \cos L_1\sin L_2 - \cos L_2\sin L_1\cos(\Omega_2 - \Omega_1) \end{matrix} \right.$$

$$\left. \begin{matrix} -\cos L_1\sin(\Omega_2 - \Omega_1) \\ \cos L_2\sin L_1 - \cos L_1\sin L_2\cos(\Omega_2 - \Omega_1) \\ \sin L_1\sin L_2 + \cos L_1\cos L_2\cos(\Omega_2 - \Omega_1) \end{matrix} \right]$$

(Note

$$\begin{bmatrix} \cos(\Omega_2 - \Omega_1) & 0 & -\sin(\Omega_2 - \Omega_1) \\ 0 & 1 & 0 \\ \sin(\Omega_2 - \Omega_1) & 0 & \cos(\Omega_2 - \Omega_1) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Omega_1 \cos\Omega_2 + \sin\Omega_1 \sin\Omega_2 & 0 & -\cos\Omega_1 \sin\Omega_2 + \cos\Omega_2 \sin\Omega_1 \\ 0 & 1 & 0 \\ \cos\Omega_1 \sin\Omega_2 - \cos\Omega_2 \sin\Omega_1 & 0 & \cos\Omega_1 \cos\Omega_2 + \sin\Omega_1 \sin\Omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Omega_2 & 0 & -\sin\Omega_2 \\ 0 & 1 & 0 \\ \sin\Omega_2 & 0 & \cos\Omega_2 \end{bmatrix} \begin{bmatrix} \cos\Omega_1 & 0 & \sin\Omega_1 \\ 0 & 1 & 0 \\ -\sin\Omega_1 & 0 & \cos\Omega_1 \end{bmatrix}$$

so that the rotation $T_{ENU(1)2ENU(2)}$ is a rotation from the first coordinates down to ECI and then up to the second coordinates.) We have

$$T_{ENU(2)2ENU(1)} = T'_{ENU(1)2ENU(2)}$$

The position vector from the ENU(1) to the ENU(2) coordinate axes (in ECI coordinates) is

$$P_{ENU(1)2ENU(2),ECI} = r_{ee} \frac{\begin{bmatrix} \cos L_2 \cos\Omega_2 \\ \cos L_2 \sin\Omega_2 \\ \left(1 - e^2\right) \sin L_2 \end{bmatrix}}{\sqrt{1 - e^2 \sin^2 L_2}} - r_{ee} \frac{\begin{bmatrix} \cos L_1 \cos\Omega_1 \\ \cos L_1 \sin\Omega_1 \\ \left(1 - e^2\right) \sin L_1 \end{bmatrix}}{\sqrt{1 - e^2 \sin^2 L_1}}$$

and, in the other coordinates, this vector is

$$P_{ENU(1)2ENU(2),ENU(i)} = T_{ECI2ENU(i)} P_{ENU(1)2ENU(2),ECI}$$

The total position coordinate transformation, including translation, can be represented by

$$P_{ENU(2)} = -P_{ENU(1)2ENU(2),ENU(2)} + T_{ENU(1)2ENU(2)} P_{ENU(1)}$$

The total velocity coordinate transformation is given by the rotation alone.

APPENDIX B:

SOLUTION TO THE CUBIC EQUATION

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In this appendix, we present the solution to the cubic polynomial equation and apply the solution to the cubic equation that arises from our tracking problem. But first, we consider the stability of our filter. The closed loop state transition matrix is

$$\Phi_{CL} = \Phi - KH = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-\alpha & T \\ -\beta/T & 1 \end{bmatrix} ;$$

hence, the eigenvalues of Φ_{CL} are

$$e_1 = 1 - \frac{1}{2}\alpha - \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$$

and

$$e_2 = 1 - \frac{1}{2}\alpha + \frac{1}{2}\sqrt{\alpha^2 - 4\beta} \ .$$

For the stability of the filter, we need the absolute value of e_1, e_2 less than one. If $\beta \leq 0$, no matter what α is, at least one of e_1, e_2 will have an absolute value of one or more; hence, a requirement for the stability of the filter is $\beta > 0$. This is the same story if $\alpha \leq 0$. If $\alpha \geq 2$ then $(1 - \alpha/2) \leq -1$ and the radical causes either the absolute value e_1 or e_2 to be one or greater. Also, for this case, depending on β, if the radical is complex, both e_1 and e_2 would have absolute value greater than one. In summary, necessary conditions for stability are:

i. $0 < \alpha < 2$

ii. $0 < \beta$.

Mentioned in Chapter 3 was the noise reduction factor, which is the ratio of the steady-state position covariance and the measurement noise intensity; see Bar-Shalom, Li, and Kirubarajan [B-1]. For this problem the noise reduction factor works out as α. For the filter to produce noise reduction, which means the noise reduction factor must be less than 1, we add the condition:

iii. $\alpha < 1$.

The canonical form for the cubic equation is [B-2]

$$y^3 + py + q = 0. \tag{B-1}$$

While equation (B-1) seems to be less general than

$$z^3 + az^2 + bz + c = 0 \ ; \tag{B-2}$$

it is not since (B-2) may be reduced to (B-1) using the transformation

$$y - \frac{a}{3} = z \ . \tag{B-3}$$

Substituting (B-3) into (B-2), we obtain

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c$$

$$= \left(y^3 - ay^2 + \frac{1}{3}a^2 y - \frac{1}{27}a^3\right) + a\left(y^2 - \frac{2}{3}ay + \frac{1}{9}a^2\right) + b\left(y - \frac{a}{3}\right) + c$$

$$= y^3 + \left(b - \frac{1}{3}a^2\right)y + \left(\frac{2}{27}a^3 - \frac{ab}{3} + c\right) \ .$$

Letting

$$p = b - \frac{a^2}{3} \tag{B-4}$$

and

$$q = \frac{2a^3}{27} - \frac{ab}{3} + c \ , \tag{B-5}$$

we reduce (B-2) to (B-1).

Returning to (B-1), define

$$w - \frac{p}{3w} = y \ . \tag{B-6}$$

Substituting (B-6) into (B-1) we have

$$\left(w - \frac{p}{3w}\right)^3 + p\left(w - \frac{p}{3w}\right) + q$$

$$= \left(w^3 - pw + \frac{1}{3}\frac{p^2}{w} - \frac{1}{27}\frac{p^3}{w^3}\right) + p\left(w - \frac{p}{3w}\right) + q$$

$$= w^3 - \frac{1}{27}\frac{p^3}{w^3} + q = 0 \ .$$

Multiplying through by w^3, we get

$$w^6 + qw^3 - \frac{p^3}{27} = 0 \ . \tag{B-7}$$

Equation (B-7) is quadratic in w^3, which has the solution (in terms of w^3)

$$w^3 = -\frac{q}{2} \pm \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} \ . \tag{B-8}$$

Taking the cube root of (B-8) we get

$$w = \sqrt[3]{-\frac{q}{2} \pm \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}} \ . \tag{B-9}$$

The cube root is a triple values function and we use the root which results in the α satisfying the three conditions mentioned in the top paragraphs. Back-substituting w into (B-6) to obtain y, and the y into (B-3), we obtain z, the solution to our cubic equation (B-2).

Our cubic polynomial of interest is (3.92). Repeating,

$$\rho^2\left(8\alpha\beta - 8\beta + 8\alpha^2 - 4\alpha^3 + 2\beta^2 - \alpha\beta^2 - 4\alpha^2\beta\right) - 8\beta^3 = 0 \ . \tag{B-10}$$

Equation (B-10) is a cubic polynomial equation in α and β. We present the solutions for α in terms of β and β in terms of α. First, we solve $\beta = \beta(\alpha)$, which is usually desired in practice.

We rewrite (B-10) as

$$\beta^3+\rho^2\left(\frac{\alpha}{8}-\frac{1}{4}\right)\beta^2+\rho^2\left(1-\alpha+\frac{\alpha^2}{2}\right)\beta+\rho^2\left(\frac{\alpha^3}{2}-\alpha^2\right)=0\ . \tag{B-11}$$

Equation (B-11) is in the form of (B-2), so we use (B-4) and (B-5) to reduce it into the form of (B-1). We have

$$p=\rho^2\left(1-\alpha+\frac{\alpha^2}{2}\right)-\frac{1}{3}\left(\rho^2\left(\frac{\alpha}{8}-\frac{1}{4}\right)\right)^2 \tag{B-12}$$

and

$$q=\frac{2}{27}\rho^6\left(\frac{\alpha}{8}-\frac{1}{4}\right)^3-\frac{1}{3}\rho^4\left(\frac{\alpha}{8}-\frac{1}{4}\right)\left(1-\alpha+\frac{\alpha^2}{2}\right)+\rho^2\left(\frac{\alpha^3}{2}-\alpha^2\right)\ . \tag{B-13}$$

Then, combining (B-3) and (B-6), we get

$$\beta=w-\frac{p}{3w}-\frac{a}{3} \tag{B-14}$$

with w given by (B-9), p given by (B-12), and by comparing (B-11) with (B-2)

$$a=\rho^2\left(\frac{\alpha}{8}-\frac{1}{4}\right)\ .$$

Equation (B-10) is a cubic in both α and β. Having solved β in terms of α, we solve for α in terms of β, $\alpha=\alpha(\beta)$. Dividing (B-10) by $-4\rho^2$ we obtain

$$\alpha^3+(\beta-2)\,\alpha^2+\left(\frac{1}{4}\beta^2-2\beta\right)\alpha+2\frac{\beta^3}{\rho^2}-\frac{1}{2}\beta^2+2\beta=0\ .$$

We have

$$p=\left(\frac{1}{4}\beta^2-2\beta\right)-\frac{1}{2}(\beta-2)^2 \tag{B-15}$$

and

$$q=\frac{2}{27}(\beta-2)^3-\frac{1}{3}(\beta-2)\left(\frac{1}{4}\beta^2-2\beta\right)+\left(2\frac{\beta^3}{\rho^2}-\frac{1}{2}\beta^2+2\beta\right)\ .$$

Referring to (B-14), we then find that

$$\alpha=w-\frac{p}{3w}-\frac{a}{3}$$

with w given by (B-9), p given by (B-15), and with

$$a=(\beta-2)\ .$$

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[B-1] Y. Bar-Shalom, X. Rong Li, and T. Kirubarajan, *Estimation with Applications to Tracking and Navigation*, John Wiley, 2001.

[B-2] J.E. Gray and W.J. Murray, "A Derivation of an Analytic Expression for the Tracking Index for the Alpha-Beta-Gamma Filter," *IEEE Transactions on Aerospace and ElectronicSystems*, Vol. 29, No. 3, July 1993.

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